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Investigation of the D_{s1} structure via B_c to $D_{s1}l^+l^-/\nu\bar{\nu}$ transitions in QCD

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Abstract

We investigate the structure of the $D_{s1}(2460, 2536)(J^P = 1^+)$ mesons via analyzing the semileptonic $B_c \to D_{s1}l^+l^-, l = \tau, \mu, e$ and $B_c \to D_{s1}\nu\bar{\nu}$ transitions in the framework of the three-point QCD sum rules. We consider the D_{s1} meson in two ways, the pure $|c\bar{s}\rangle$ state and then as a mixture of two $|^3P_1\rangle$ and $|^1P_1\rangle$ states. Such types of rare transitions take place at loop level by electroweak penguin and weak box diagrams in the standard model via the flavor changing neutral current transition of $b \to s$. The relevant form factors are calculated taking into account the gluon condensate contributions. These form factors are numerically obtained for $|c\bar{s}\rangle$ case and plotted in terms of the unknown mixing angle θ_s, when the D_{s1} meson is considered as a mixture of two $|^3P_1\rangle$ and $|^1P_1\rangle$ states. The obtained results for the form factors are used to evaluate the decay rates and branching ratios. Any future experimental measurement on these form factors as well as decay rates and branching fractions and their comparison with the obtained results in the present work can give considerable information about the structure of this meson and the mixing angle θ_s.

1. Introduction

The structure of the even-parity charmed D_{sJ} mesons is not known exactly yet and has been debated in the quark model. The observation of two narrow resonances with charm and strangeness, $D_{s0}(2317)$ in the invariant mass distribution of $D_s\pi^0$ [1–6] and $D_{s1}(2460)$ in the $D_s^*\pi^0$ and $D_s\gamma$ mass distributions [3–8], has raised discussions about the structure of these states and their quark contents [9, 10]. Analysis of $D_{s_0}(2317) \to D_s^*\gamma$, $D_{s1}(2460) \to D_s^*\gamma$ and $D_{s_1}(2460) \to D_{s_0}(2317)\gamma$ shows that the quark content of these mesons is probably $c\bar{s}$ [11]. Among these mesons, the axial vector charm-strange meson D_{s1} is the more

Table 1. Masses of 1^1P_1 and 1^3P_1 heavy–light mesons in quark models.

References	[12]	[13]	[14]
$D_{s1}1(^3P_1)$	2.57	2.55	2.535
$D_{s1}2(^1P_1)$	2.53	2.55	2.605

attractive one, because the discovery of the $D_{s1}(2460)(J^P = 1^+)$ meson [2–5] and its measured mass indicated a lower mass than expected in the potential model (PM) [12] and the quark model (QM) [13, 14] predictions. In other words, $D_{s1}(2460)$ does not fit easily into the $c\bar{s}$ spectroscopy [15]. However, some physicists presumed that this discovered state is a conventional $c\bar{s}$ meson [16–26]. Many different theoretical efforts have been dedicated to the understanding of this unexpected and surprising disparity between theory and experiment [27–35]. As a result of the above discussion, we will consider the D_{s1} meson in two ways, the pure $|c\bar{s}\rangle$ state and also as a mixture of two $|^3P_1\rangle$ and $|^1P_1\rangle$ states.

Heavy–light mesons are not charge conjugation eigen states and so mixing can occur among states with the same J^P and different mass that are forbidden for neutral states. These occur between states with $J = L$ and $S = 1$ or 0 [15]. Hence, the mixing of the physical D_{s1} and D'_{s1} states can be parameterized in terms of a mixing angle θ_s, as follows:

$$|D_{s1}\rangle = \sin\theta_s|^3P_1\rangle + \cos\theta_s|^1P_1\rangle, \qquad |D'_{s1}\rangle = \cos\theta_s|^3P_1\rangle - \sin\theta_s|^1P_1\rangle, \tag{1}$$

where the spectroscopic notation $^{2S+1}L_J$ has been used to introduce the mixing states. Considering $|^3P_1\rangle \equiv |D_{s1}1\rangle$ and $|^1P_1\rangle \equiv |D_{s1}2\rangle$ with different masses and decay constants [36], we can apply these relations for axial vectors $D_{s1}(2460)$ and $D_{s1}(2536)$ mesons with two different masses, i.e.,

$$\begin{aligned} |D_{s1}(2460)\rangle &= \sin\theta_s|D_{s1}1\rangle + \cos\theta_s|D_{s1}2\rangle, \\ |D_{s1}(2536)\rangle &= \cos\theta_s|D_{s1}1\rangle - \sin\theta_s|D_{s1}2\rangle. \end{aligned} \tag{2}$$

The masses of $D_{s1}1$ and $D_{s1}2$ states are presented in table 1. These values have been obtained in the QM approach.

Note that in the heavy quark limit, the physical eigen states D_{s1} and D'_{s1} can be identified with $P_1^{1/2}$ and $P_1^{3/2}$ with notation L_J^j, where j is the total angular momentum of the light quark [37], corresponding to $\theta_s = -54.7^\circ$ [36].

In this work, taking into account the gluon condensate corrections, we analyze the rare semileptonic $B_c \to D_{s1}l^+l^-, l = \tau, \mu, e$ and $B_c \to D_{s1}\nu\bar{\nu}$ transitions in a three-point QCD sum rules (3PSR) approach. Note that the $B_c \to (D^*, D_s^*, D_{s1}(2460))\nu\bar{\nu}$ transitions have been studied in [38], but assuming D_{s1} only as $c\bar{s}$. The $B_c \to D_q l^+l^-/\nu\bar{\nu}$ [39], $B_c \to D_q^* l^+l^-$ $(q = d, s)$ [40] transitions have also been analyzed in the same framework.

The heavy B_c meson contains two heavy quarks b and c with different charges. This meson is similar to the charmonium and bottomonium in the spectroscopy, but in contrast to the charmonium and bottomonium, B_c decays only via weak interaction and has a long lifetime. The study of the B_c transitions is useful for more precise determination of the Cabibbo, Kabayashi and Maskawa (CKM) matrix elements in the weak decays.

The rare semileptonic $B_c \to D_{s1}l^+l^-/\nu\bar{\nu}$ decays occur at loop level by electroweak penguin and weak box diagrams in the standard model (SM) via the flavor changing neutral current (FCNC) transition of $b \to sl^+l^-$. The FCNC decays of the B_c meson are sensitive to new physics (NP) contributions to penguin operators. Therefore, the study of such FCNC transitions can improve the information about

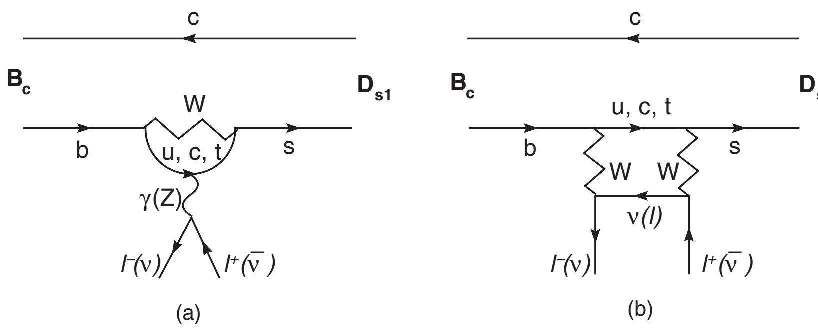

Figure 1. The loop diagrams of the semileptonic decay of B_c to D_{s1}. The electroweak penguin and box diagrams are shown in parts (a) and (b), respectively.

- the CP violation, T violation and polarization asymmetries in $b \to s$ penguin channels, that occur in weak interactions;
- new operators or operators that are subdominant in the SM;
- establishing NP and flavor physics beyond the SM.

To obtain the form factors of the semileptonic $B_c \to D_{s1}(2460[2536])$ transitions, first, we will suppose the $D_{s1}(2460)$ and $D_{s1}(2536)$ axial vector mesons as the pure $|c\bar{s}\rangle$ state and calculate the related form factors. Second, we will consider the D_{s1} meson as a mixture of two components $|D_{s1}1\rangle$ and $|D_{s1}2\rangle$ states and calculate the form factors of the $B_c \to D_{s1}1$ and $B_c \to D_{s1}2$ transitions. With the help of equation (2) and the definition of the form factors which will be presented in the following section, we will derive the transition form factors of $B_c \to D_{s1}(2460[2536])$ decays as a function of the mixing angle θ_s. The future experimental study of such rare decays and comparison of the results with the predictions of theoretical calculations can improve the information about the structure of the D_{s1} meson and the mixing angle θ_s.

This paper is organized as follow. In section 2, we calculate the form factors for the $B_c \to D_{s1}$ transition in the 3PSR. In section 3, the two-gluon condensate contributions as non-perturbative corrections are calculated. The calculation of the decay rates for $B_c \to D_{s1}l^+l^-$ and $B_c \to D_{s1}\nu\bar{\nu}$ transitions is presented in section 4. Finally, section 5 is devoted to the numeric results and discussions.

2. The form factors of $B_c \to D_{s1}$ transition in 3PSR

In the standard model, the effective Hamiltonian responsible for the rare semileptonic $B_c \to D_{s1}l^+l^-$ and $B_c \to D_{s1}\nu\bar{\nu}$ decays, which are described via $b \to sl^+l^-$ loop transitions (see figure 1) at quark level, can be written as

$$\mathcal{H}_{\text{eff}} = \frac{G_F\alpha}{2\pi\sqrt{2}}V_{tb}V_{ts}^*\bigg[C_9^{\text{eff}}\bar{s}\gamma_\mu(1-\gamma_5)b\bar{\ell}\gamma_\mu\ell + C_{10}\bar{s}\gamma_\mu(1-\gamma_5)b\bar{\ell}\gamma_\mu\gamma_5\ell - 2C_7^{\text{eff}}\frac{m_b}{q^2}\bar{s}\,\mathrm{i}\sigma_{\mu\nu}q^\nu(1+\gamma_5)b\bar{\ell}\gamma_\mu\ell\bigg], \tag{3}$$

where C_7^{eff}, C_9^{eff} and C_{10} are the Wilson coefficients, G_F is the Fermi constant, α is the fine structure constant at the Z mass scale and V_{ij} are the elements of the CKM matrix.

These loop transitions occur via the intermediate u, c, t quarks. In the SM, the measurement of the forward–backward asymmetry and invariant dilepton mass distribution in

$b \to q' l^+ l^-$ $(q' = d, s)$ transitions provide information on the short distance contributions dominated by the top quark loops [41]. The electroweak penguin involving the contributions of photon and Z bosons shown in figure 1(a) and figure 1(b) presents the contribution of the W box diagram. It is recalled that the $b \to s\nu\bar{\nu}$ transition receives contributions only from Z-penguin and box diagrams.

The transition amplitude of $B_c \to D_{s1} l^+ l^- / \nu\bar{\nu}$ decays is obtained sandwiching equation (3) between the initial and final states, i.e.,

$$\begin{aligned}\mathcal{M} = \frac{G_F \alpha}{2\pi\sqrt{2}} V_{tb} V_{ts}^* \bigg[& C_9^{\text{eff}} \langle D_{s1}(p') | \bar{s}\gamma_\mu (1-\gamma_5) b | B_c(p) \rangle \bar{\ell}\gamma_\mu \ell \\ & + C_{10} \langle D_{s1}(p') | \bar{s}\gamma_\mu (1-\gamma_5) b | B_c(p) \rangle \bar{\ell}\gamma_\mu\gamma_5 \ell \\ & - 2C_7^{\text{eff}} \frac{m_b}{q^2} \langle D_{s1}(p') | \bar{s} i\sigma_{\mu\nu} q^\nu (1+\gamma_5) b | B_c(p) \rangle \bar{\ell}\gamma_\mu \ell \bigg], \end{aligned} \tag{4}$$

where p and p' are the momentum of initial and final meson states, respectively, and ε is the polarization vector of the D_{s1} meson. Our aim is to parameterize the matrix elements appearing in equation (4) in terms of the transition form factors considering the Lorentz invariance and parity considerations

$$\begin{aligned} \langle D_{s1}(p',\varepsilon) | \bar{s}\gamma_\mu\gamma_5 b | B_c(p) \rangle &= \frac{2A_V^{B_c\to D_{s1}}(q^2)}{m_{B_c} + m_{D_{s1}}} \varepsilon_{\mu\nu\alpha\beta} \varepsilon^{*\nu} p^\alpha p'^\beta, \\ \langle D_{s1}(p',\varepsilon) | \bar{s}\gamma_\mu b | B_c(p) \rangle &= -\,\mathrm{i} A_0^{B_c\to D_{s1}}(q^2) \big(m_{B_c} + m_{D_{s1}}\big) \varepsilon_\mu^* \\ & \quad + \mathrm{i} \frac{A_1^{B_c\to D_{s1}}(q^2)}{m_{B_c} + m_{D_{s1}}} (\varepsilon^* p) P_\mu + \mathrm{i} \frac{A_2^{B_c\to D_{s1}}(q^2)}{m_{B_c} + m_{D_{s1}}} (\varepsilon^* p) q_\mu, \\ \langle D_{s1}(p',\varepsilon) | \bar{s}\sigma_{\mu\nu} q^\nu \gamma_5 b | B_c(p) \rangle &= 2T_V^{B_c\to D_{s1}}(q^2)\, \mathrm{i}\varepsilon_{\mu\nu\alpha\beta} \varepsilon^{*\nu} p^\alpha p'^\beta, \\ \langle D_{s1}(p',\varepsilon) | \bar{s}\sigma_{\mu\nu} q^\nu b | B_c(p) \rangle &= T_0^{B_c\to D_{s1}}(q^2) \big[\varepsilon_\mu^* \big(m_{B_c}^2 - m_{D_{s1}}^2\big) - (\varepsilon^* p) P_\mu\big] \\ & \quad + T_1^{B_c\to D_{s1}}(q^2)(\varepsilon^* p) \bigg[q_\mu - \frac{q^2}{m_{B_c}^2 - m_{D_{s1}}^2} P_\mu \bigg], \end{aligned} \tag{5}$$

where $A_i^{B_c\to D_{s1}}(q^2), i = V, 0, 1, 2$ and $T_j^{B_c\to D_{s1}}(q^2), j = V, 0, 1$ are the transition form factors, $P_\mu = (p + p')_\mu$ and $q_\mu = (p - p')_\mu$. Here, q^2 is the momentum transfer squared of the Z boson (photon). In order that our calculations are simple, the following redefinitions of the transition form factors are considered:

$$\begin{aligned} A_V'^{B_c\to D_{s1}}(q^2) &= \frac{2A_V^{B_c\to D_{s1}}(q^2)}{m_{B_c} + m_{D_{s1}}}, & A_0'^{B_c\to D_{s1}}(q^2) &= A_0^{B_c\to D_{s1}}(q^2)\big(m_{B_c} + m_{D_{s1}}\big), \\ A_1'^{B_c\to D_{s1}}(q^2) &= -\frac{A_1^{B_c\to D_{s1}}(q^2)}{m_{B_c} + m_{D_{s1}}}, & A_2'^{B_c\to D_{s1}}(q^2) &= -\frac{A_2^{B_c\to D_{s1}}(q^2)}{m_{B_c} + m_{D_{s1}}}, \\ T_V'^{B_c\to D_{s1}}(q^2) &= -2T_V^{B_c\to D_{s1}}(q^2), & T_0'^{B_c\to D_{s1}}(q^2) &= -T_0^{B_c\to D_{s1}}(q^2)\big(m_{B_c}^2 - m_{D_{s1}}^2\big), \\ T_1'^{B_c\to D_{s1}}(q^2) &= -T_1^{B_c\to D_{s1}}(q^2). \end{aligned} \tag{6}$$

To calculate the form factors within the three-point QCD sum rules method, the following three-point correlation functions are used:

$$\begin{aligned} \Pi_{\mu\nu}^{V-A}(p^2, p'^2, q^2) &= \mathrm{i}^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{e}^{-\mathrm{i}px}\, \mathrm{e}^{\mathrm{i}p'y} \langle 0 | T\big[J_\nu^{D_{s1}}(y) J_\mu^{V-A}(0) J^{B_c\dagger}(x)\big] | 0 \rangle, \\ \Pi_{\mu\nu}^{T-PT}(p^2, p'^2, q^2) &= \mathrm{i}^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{e}^{-\mathrm{i}px}\, \mathrm{e}^{\mathrm{i}p'y} \langle 0 | T\big[J_\nu^{D_{s1}}(y) J_\mu^{T-PT}(0) J^{B_c\dagger}(x)\big] | 0 \rangle, \end{aligned} \tag{7}$$

where $J_\nu^{D_{s1}}(y) = \bar{c}\gamma_\nu\gamma_5 s$ and $J^{B_c}(x) = \bar{c}\gamma_5 b$ are the interpolating currents of the initial and final meson states, respectively. $J_\mu^{V-A} = \bar{s}\gamma_\mu(1-\gamma_5)b$ and $J_\mu^{T-PT} = \bar{s}\sigma_{\mu\nu}q^\nu(1+\gamma_5)b$ are the vector–axial vector and tensor–pseudo tensor parts of the transition currents. In QCD sum rules approach, we can obtain the correlation function of equation (7) in two sides. The phenomenological or physical part is calculated saturating the correlator by a tower of hadrons with the same quantum numbers as interpolating currents. The QCD or theoretical part, on the other side, is obtained in terms of the quarks and gluons interacting in the QCD vacuum. To drive the phenomenological part of the correlators given in equation (7), two complete sets of intermediate states with the same quantum numbers as the currents $J_{D_{s1}}$ and J_{B_c} are inserted. This procedure leads to the following representations of the above-mentioned correlators:

$$\begin{aligned}\Pi_{\mu\nu}^{V-A}(p^2,p'^2,q^2) &= \frac{\langle 0|J_\nu^{D_{s1}}|D_{s1}(p',\varepsilon)\rangle\langle D_{s1}(p',\varepsilon)|J_\mu^{V-A}|B_c(p)\rangle\langle B_c(p)|J^{B_c\dagger}|0\rangle}{\left(p'^2-m_{D_{s1}}^2\right)\left(p^2-m_{B_c}^2\right)}\\ &\quad+\text{higher resonances and continuum states},\\ \Pi_{\mu\nu}^{T-PT}(p^2,p'^2,q^2) &= \frac{\langle 0|J_\nu^{D_{s1}}|D_{s1}(p',\varepsilon)\rangle\langle D_{s1}(p',\varepsilon)|J_\mu^{T-PT}|B_c(p)\rangle\langle B_c(p)|J^{B_c\dagger}|0\rangle}{\left(p'^2-m_{D_{s1}}^2\right)\left(p^2-m_{B_c}^2\right)}\\ &\quad+\text{higher resonances and continuum states}.\end{aligned} \tag{8}$$

The following matrix elements are defined in the standard way in terms of the leptonic decay constants of the D_{s1} and B_c mesons as

$$\langle 0|J_{D_{s1}}^\nu|D_{s1}(p',\varepsilon)\rangle = f_{D_{s1}}m_{D_{s1}}\varepsilon^\nu, \qquad \langle 0|J_{B_c}|B_c(p)\rangle = \mathrm{i}\frac{f_{B_c}m_{B_c}^2}{m_b+m_c}. \tag{9}$$

Using equations (5), (6) and (9) in equation (8) and performing summation over the polarization of the D_{s1} meson we obtain

$$\begin{aligned}\Pi_{\mu\nu}^{V-A}(p^2,p'^2,q^2) &= -\frac{f_{B_c}m_{B_c}^2}{(m_b+m_c)}\frac{f_{D_{s1}}m_{D_{s1}}}{\left(p'^2-m_{D_{s1}}^2\right)\left(p^2-m_{B_c}^2\right)}\times\left[\mathrm{i}A_V'^{B_c\to D_{s1}}(q^2)\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta\right.\\ &\quad\left.+A_0'^{B_c\to D_{s1}}(q^2)g_{\mu\nu}+A_1'^{B_c\to D_{s1}}(q^2)P_\mu p_\nu+A_2'^{B_c\to D_{s1}}(q^2)q_\mu p_\nu\right]+\text{excited states},\\ \Pi_{\mu\nu}^{T-PT}(p^2,p'^2,q^2) &= -\frac{f_{B_c}m_{B_c}^2}{(m_b+m_c)}\frac{f_{D_{s1}}m_{D_{s1}}}{\left(p'^2-m_{D_{s1}}^2\right)\left(p^2-m_{B_c}^2\right)}\times\left[T_V'^{B_c\to D_{s1}}(q^2)\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta\right.\\ &\quad\left.-\mathrm{i}T_0'^{B_c\to D_{s1}}(q^2)g_{\mu\nu}-\mathrm{i}T_1'^{B_c\to D_{s1}}(q^2)q_\mu p_\nu\right]+\text{excited states}.\end{aligned} \tag{10}$$

To calculate the form factors, A'_V, A'_0, A'_1, A'_2, T'_V, T'_0 and T'_1, we will choose the structures, $\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta$, $g_{\mu\nu}$, $P_\mu p_\nu$, $q_\mu p_\nu$, from $\Pi_{\mu\nu}^{V-A}$ and $\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta$, $\mathrm{i}g_{\mu\nu}$ and $\mathrm{i}q_\mu p_\nu$ from $\Pi_{\mu\nu}^{T-PT}$, respectively.

On the QCD side, using the operator product expansion (OPE), we can obtain the correlation function in quark–gluon language in the deep Euclidean region where $p^2 \ll (m_b+m_c)^2$ and $p'^2 \ll (m_c^2+m_s^2)$. For this aim, the correlators are written as

$$\begin{aligned}\Pi_{\mu\nu}^{V-A}(p^2,p'^2,q^2) &= \mathrm{i}\Pi_V^{V-A}\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta+\Pi_0^{V-A}g_{\mu\nu}+\Pi_1^{V-A}P_\mu p_\nu+\Pi_2^{V-A}q_\mu p_\nu,\\ \Pi_{\mu\nu}^{T-PT}(p^2,p'^2,q^2) &= \Pi_V^{T-PT}\varepsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta-\mathrm{i}\Pi_0^{T-PT}g_{\mu\nu}-\mathrm{i}\Pi_1^{T-PT}q_\mu p_\nu,\end{aligned} \tag{11}$$

where each Π_i function is defined in terms of the perturbative and non-perturbative parts as

$$\Pi_i(p^2,p'^2,q^2) = \Pi_i^{\mathrm{per}}(p^2,p'^2,q^2)+\Pi_i^{\mathrm{nonper}}(p^2,p'^2,q^2). \tag{12}$$

To obtain the perturbative part of the correlation function, we should study the bare loop diagrams in figure 1. In calculating the bare loop contributions, we first write the double dispersion representation for the coefficients of the corresponding Lorentz structures appearing in each correlation function, as follows:

$$\Pi_i^{\text{per}} = -\frac{1}{(2\pi)^2}\int ds'\int ds\,\frac{\rho_i(s,s',q^2)}{(s-p^2)(s'-p'^2)} + \text{subtraction terms.} \tag{13}$$

The spectral densities $\rho_i^{\text{per}}(s,s',q^2)$ are calculated by the help of the Gutkosky rules, i.e., the propagators are replaced by Dirac–delta functions

$$\frac{1}{p^2-m^2} \to -2\mathrm{i}\pi\,\delta(p^2-m^2), \tag{14}$$

expressing that all quarks are real. Note that there are two main vertices related to the bare loop diagrams that describe $b \to sl^+l^-$ transition in figure 1, i.e., $\gamma_\mu(1-\gamma_5)$ and $\sigma_{\mu\nu}q^\nu(1+\gamma_5)$. First, we calculate the spectral densities related to $\gamma_\mu(1-\gamma_5)$ vertex. Straightforward calculations end up in the following results:

$$\begin{aligned}
\rho_V^{V-A} &= 4N_c I_0(s,s',q^2)\,\{B_1(m_b-m_c) - B_2(m_s+m_c) - m_c\}\,,\\
\rho_0^{V-A} &= -2N_c I_0(s,s',q^2)\{\Delta(m_c+m_s) - \Delta'(m_b-m_c) - 4A_1(m_b-m_c)\\
&\quad + 2m_c^2(m_b-m_c-m_s) + m_c(2m_bm_s-u)\},\\
\rho_1^{V-A} &= 2N_c I_0(s,s',q^2)\{B_1(m_b-3m_c) - B_2(m_c+m_s) + 2A_2(m_b-m_c)\\
&\quad + 2A_3(m_b-m_c) - m_c\},\\
\rho_2^{V-A} &= 2N_c I_0(s,s',q^2)\{2A_2(m_b-m_c) - 2A_3(m_b-m_c) - B_1(m_b+m_c)\\
&\quad + B_2(m_c+m_s) + m_c\}.
\end{aligned} \tag{15}$$

Then, the spectral densities related to the $\sigma_{\mu\nu}q^\nu(1+\gamma_5)$ vertex are presented as

$$\begin{aligned}
\rho_V^{T-PT}(s,s',q^2) &= 2N_c I_0(s,s',q^2)\big\{m_c(m_b-m_s) + B_1\big[\Delta - m_cm_s - m_c^2\\
&\quad + m_b(m_c+m_s) - s\big] + B_2\big[\Delta' - m_cm_s - m_c^2 + m_b(m_c+m_s) - s'\big]\big\},\\
\rho_0^{T-PT}(s,s',q^2) &= 2N_c I_0(s,s',q^2)\{2A_1(2s-u) - \Delta[m_b(m_c+m_s) - m_c(m_c+m_s) + s']\\
&\quad + \Delta'[m_b(m_c+m_s) - m_c(m_c+m_s) + s] - 2m_c(m_b-m_c)s'\\
&\quad - 2m_c(m_c+m_s)s + 2m_c(m_b+m_s)u\},\\
\rho_1^{T-PT}(s,s',q^2) &= 2N_c I_0(s,s',q^2)\{m_c(m_b+m_s) + B_2[(m_b-m_c)(m_c+m_s) + s']\\
&\quad + B_1[\Delta - \Delta' - m_b(m_c+m_s) + m_c(m_c+m_s) - s] + (A_2-A_3)(2s-u)\},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
I_0(s,s',q^2) &= \frac{1}{4\lambda^{1/2}(s,s',q^2)},\\
\lambda(a,b,c) &= a^2+b^2+c^2-2ac-2bc-2ab,\\
\Delta' &= \left(s'+m_c^2-m_s^2\right),\\
\Delta &= \left(s+m_c^2-m_b^2\right),\\
u &= s+s'-q^2,\\
B_1 &= \frac{1}{\lambda(s,s',q^2)}[2s'\Delta - \Delta' u],\\
B_2 &= \frac{1}{\lambda(s,s',q^2)}[2s\Delta' - \Delta u],
\end{aligned}$$

$$A_1 = -\frac{1}{2\lambda(s, s', q^2)}\big[\big(4ss'm_c^2 - s\Delta'^2 - s'\Delta^2 - u^2m_c^2 + u\Delta\Delta'\big)\big],$$
$$A_2 = -\frac{1}{\lambda^2(s, s', q^2)}\big[8ss'^2m_c^2 - 2ss'\Delta'^2 - 6s'^2\Delta^2 - 2u^2s'm_c^2 + 6s'u\Delta\Delta' - u^2\Delta'^2\big],$$
$$A_3 = \frac{1}{\lambda^2(s, s', q^2)}\big[4ss'um_c^2 + 4ss'\Delta\Delta' - 3su\Delta'^2 - 3u\Delta^2s' - u^3m_c^2 + 2u^2\Delta\Delta'\big].$$

and $N_c = 3$ is the color factor.

The integration region in equation (13) is obtained requiring that the argument of three delta vanishes, simultaneously. The physical region in the s- and s'-plane is described by the following inequalities:

$$-1 \leqslant \frac{2ss' + (s + s' - q^2)\big(m_b^2 - s - m_c^2\big) + \big(m_c^2 - m_s^2\big)2s}{\lambda^{1/2}\big(m_b^2, s, m_c^2\big)\lambda^{1/2}(s, s', q^2)} \leqslant +1. \tag{17}$$

From this inequality, to use in the lower limit of the integration over s in continuum subtractions, it is easy to express s in terms of s', i.e., s_L is as follows:

$$s_L = \frac{\big(m_c^2 + q^2 - m_b^2 - s'\big)\big(m_b^2s' - q^2m_c^2\big)}{\big(m_b^2 - q^2\big)\big(m_c^2 - s'\big)}. \tag{18}$$

3. Gluon condensate contribution

In this section, the non-perturbative part contributions to the correlation function are discussed. Here, we will follow the same procedure as stated in [38–40, 42]. The non-perturbative part contains the quark and gluon condensate diagrams. For this aim, we consider the condensate terms of dimensions 3, 4 and 5. It is found that the heavy quark condensate contributions are suppressed by the inverse of the heavy quark mass and can be safely omitted. The light s quark condensate contribution is zero after applying the double Borel transformation with respect to the both variables p^2 and p'^2, because only one variable appears in the denominator. Therefore, in this case, we consider the two gluon condensate diagrams with mass dimension 4 as non-perturbative corrections. The diagrams for the contribution of the gluon condensates are depicted in figure 2. To obtain the contributions of these diagrams, the Fock–Schwinger fixed-point gauge, $x^\mu A_\mu^a = 0$, are used, where A_μ^a is the gluon field. In the evaluation of the diagrams in figure 2, integrals of the following types are encountered:

$$I_0(a, b, c) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{\big[k^2 - m_c^2\big]^a\big[(p + k)^2 - m_b^2\big]^b\big[(p' + k)^2 - m_s^2\big]^c},$$
$$I_\mu(a, b, c) = \int \frac{d^4k}{(2\pi)^4}\frac{k_\mu}{\big[k^2 - m_c^2\big]^a\big[(p + k)^2 - m_b^2\big]^b\big[(p' + k)^2 - m_s^2\big]^c}, \tag{19}$$
$$I_{\mu\nu}(a, b, c) = \int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{\big[k^2 - m_c^2\big]^a\big[(p + k)^2 - m_b^2\big]^b\big[(p' + k)^2 - m_s^2\big]^c}.$$

These integrals can be calculated using the Schwinger representation for the Euclidean propagator

$$\frac{1}{(k^2 + m^2)^n} = \frac{1}{\Gamma(n)}\int_0^\infty d\alpha\,\alpha^{n-1}\,e^{-\alpha(k^2+m^2)}. \tag{20}$$

After Borel transformation using

$$\mathcal{B}_{\hat{p}^2}(M^2)\,e^{-\alpha p^2} = \delta(1/M^2 - \alpha), \tag{21}$$

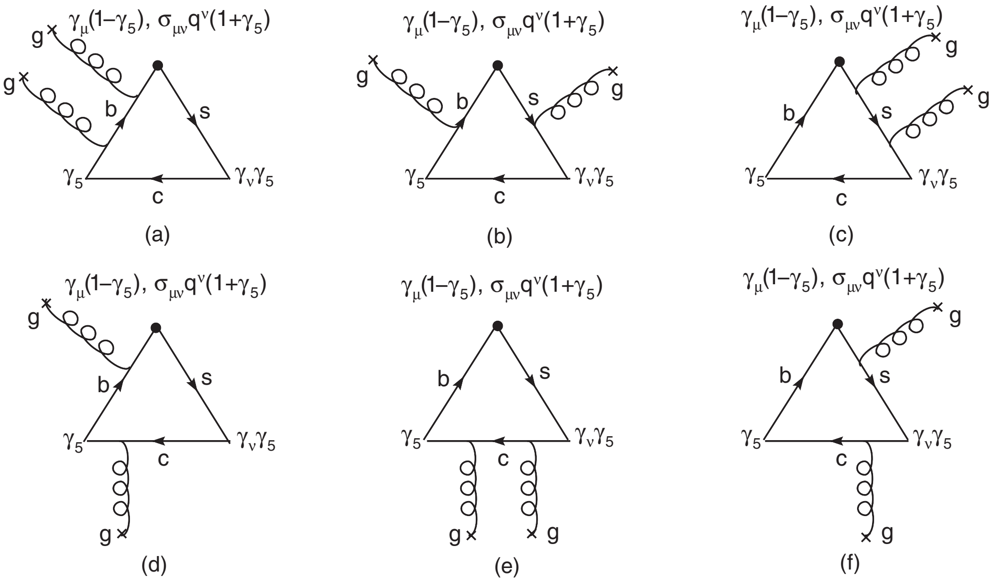

Figure 2. Contribution of gluon condensates for $B_c \to D_{s1}$ transition.

we obtain

$$\begin{aligned}
\hat{I}_0(a,b,c) &= \frac{(-1)^{a+b+c}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)}\left(M_1^2\right)^{2-a-b}(M_2^2)^{2-a-c}\mathcal{U}_0(a+b+c-4,1-c-b),\\
\hat{I}_\mu(a,b,c) &= \hat{I}_1(a,b,c)p_\mu + \hat{I}_2(a,b,c)p'_\mu,\\
\hat{I}_{\mu\nu}(a,b,c) &= \hat{I}_6(a,b,c)g_{\mu\nu} + \hat{I}_3(a,b,c)p_\mu p_\nu + \hat{I}_4(a,b,c)p_\mu p'_\nu\\
&\quad + \hat{I}_4(a,b,c)p'_\mu p_\nu + \hat{I}_5(a,b,c)p'_\mu p'_\nu.
\end{aligned} \tag{22}$$

$\hat{I}$ in equation (22) stands for the double Borel transformed form of equation (19). In the Schwinger representation,

$$\begin{aligned}
\hat{I}_k(a,b,c) &= \mathrm{i}\frac{(-1)^{a+b+c+1}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)}\left(M_1^2\right)^{1-a-b+k}\left(M_2^2\right)^{4-a-c-k}\mathcal{U}_0(a+b+c-5,1-c-b),\\
\hat{I}_m(a,b,c) &= \mathrm{i}\frac{(-1)^{a+b+c+1}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)}\left(M_1^2\right)^{-a-b-1+m}\left(M_2^2\right)^{7-a-c-m}\mathcal{U}_0(a+b+c-5,1-c-b),\\
\hat{I}_6(a,b,c) &= \mathrm{i}\frac{(-1)^{a+b+c+1}}{32\pi^2\Gamma(a)\Gamma(b)\Gamma(c)}\left(M_1^2\right)^{3-a-b}\left(M_2^2\right)^{3-a-c}\mathcal{U}_0(a+b+c-6,2-c-b),
\end{aligned} \tag{23}$$

where $k = 1, 2, m = 3, 4, 5$, M_1^2 and M_2^2 are the Borel parameters in the s and s' channel, respectively, and the function $\mathcal{U}_0(a,b)$ is defined as

$$\mathcal{U}_0(a,b) = \int_0^\infty \mathrm{d}y\left(y+M_1^2+M_2^2\right)^a y^b \exp\left[-\frac{B_{-1}}{y} - B_0 - B_1 y\right],$$

where

$$\begin{aligned}
B_{-1} &= \frac{1}{M_1^2M_2^2}\left[m_s^2M_1^4 + m_b^2M_2^4 + M_2^2M_1^2\left(m_b^2+m_s^2-q^2\right)\right],\\
B_0 &= \frac{1}{M_1^2M_2^2}\left[\left(m_s^2+m_c^2\right)M_1^2 + M_2^2\left(m_b^2+m_c^2\right)\right],
\end{aligned}$$

$$B_1 = \frac{m_c^2}{M_1^2 M_2^2}. \tag{24}$$

Performing the double Borel transformation over the variables p^2 and p'^2 on the physical as well as perturbative parts of the correlation functions and equating the coefficients of the selected structures from both sides, the sum rules for the form factors $A_i'^{B_c\to D_{s1}}$ are obtained:

$$A_i'^{B_c\to D_{s1}} = -\frac{(m_b+m_c)}{f_{B_c} m_{B_c}^2 f_{D_{s1}} m_{D_{s1}}} \, \mathrm{e}^{\frac{m_{B_c}^2}{M_1^2}} \, \mathrm{e}^{\frac{m_{D_{s1}}^2}{M_2^2}} \left\{ -\frac{1}{4\pi^2} \int_{m_c^2}^{s_0'} \mathrm{d}s' \int_{s_L}^{s_0} \rho_i^{V-A}(s,s',q^2)\, \mathrm{e}^{\frac{-s}{M_1^2}}\, \mathrm{e}^{\frac{-s'}{M_2^2}} \right.$$
$$\left. - \mathrm{i} M_1^2 M_2^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{C_i^{V-A}}{6} \right\}, \tag{25}$$

where $i = V, 0, 1, 2$ and for the form factors $T_j'^{B_c\to D_{s1}}$, we get

$$T_j'^{B_c\to D_{s1}} = -\frac{(m_b+m_c)}{f_{B_c} m_{B_c}^2 f_{D_{s1}} m_{D_{s1}}} \, \mathrm{e}^{\frac{m_{B_c}^2}{M_1^2}} \, \mathrm{e}^{\frac{m_{D_{s1}}^2}{M_2^2}} \left\{ -\frac{1}{4\pi^2} \int_{m_c^2}^{s_0'} \mathrm{d}s' \int_{s_L}^{s_0} \rho_j^{T-PT}(s,s',q^2)\, \mathrm{e}^{\frac{-s}{M_1^2}}\, \mathrm{e}^{\frac{-s'}{M_2^2}} \right.$$
$$\left. - \mathrm{i} M_1^2 M_2^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{C_j^{T-PT}}{6} \right\}, \tag{26}$$

where $j = V, 0, 1$. s_0 and s_0' are the continuum thresholds in B_c and D_{s1} channels, respectively and lower bound s_L in the integrals is given in equation (18). We present the explicit expressions of the coefficients $C_{i(j)}^{V-A(T-PT)}$ correspond to gluon condensates in appendix A.

Now, the $A_i'^{B_c\to D_{s1}1(2)}$ and $T_j'^{B_c\to D_{s1}1(2)}$ form factors are obtained from the above equations replacing the $f_{D_{s1}}$ by decay constant $f_{D_{s1}1(2)}$, and $m_{D_{s1}}$ with $m_{D_{s1}1(2)}$, i.e.,

$$A_i'^{B_c\to D_{s1}1(2)} = -\frac{(m_b+m_c)}{f_{B_c} m_{B_c}^2 f_{D_{s1}1(2)} m_{D_{s1}1(2)}} \, \mathrm{e}^{\frac{m_{B_c}^2}{M_1^2}} \, \mathrm{e}^{\frac{m_{D_{s1}1(2)}^2}{M_2^2}} \left\{ -\frac{1}{4\pi^2} \int_{m_c^2}^{s_0'} \mathrm{d}s' \right.$$
$$\left. \times \int_{s_L}^{s_0} \rho_i^{V-A}(s,s',q^2)\, \mathrm{e}^{\frac{-s}{M_1^2}}\, \mathrm{e}^{\frac{-s'}{M_2^2}} - \mathrm{i} M_1^2 M_2^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{C_i^{V-A}}{6} \right\} \tag{27}$$

and

$$T_j'^{B_c\to D_{s1}1(2)} = -\frac{(m_b+m_c)}{f_{B_c} m_{B_c}^2 f_{D_{s1}1(2)} m_{D_{s1}1(2)}} \, \mathrm{e}^{\frac{m_{B_c}^2}{M_1^2}} \, \mathrm{e}^{\frac{m_{D_{s1}1(2)}^2}{M_2^2}} \left\{ -\frac{1}{4\pi^2} \int_{m_c^2}^{s_0'} \mathrm{d}s' \right.$$
$$\left. \times \int_{s_L}^{s_0} \rho_i^{T-PT}(s,s',q^2)\, \mathrm{e}^{\frac{-s}{M_1^2}}\, \mathrm{e}^{\frac{-s'}{M_2^2}} - \mathrm{i} M_1^2 M_2^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{C_i^{T-PT}}{6} \right\}. \tag{28}$$

Using the straightforward calculations, the form factors of $A_i^{B_c\to D_{s1}(2460)}$ and $T_j^{B_c\to D_{s1}(2460)}$ are found as follows:

$$\begin{aligned}
A_0^{B_c\to D_{s1}(2460)} &= \left(\frac{m_{B_c}+m_{D_{s1}1}}{m_{B_c}+m_{D_{s1}}}\right) A_0^{B_c\to D_{s1}1} \sin\theta_s + \left(\frac{m_{B_c}+m_{D_{s1}2}}{m_{B_c}+m_{D_{s1}}}\right) A_0^{B_c\to D_{s1}2} \cos\theta_s, \\
A_{i'}^{B_c\to D_{s1}(2460)} &= \left(\frac{m_{B_c}+m_{D_{s1}}}{m_{B_c}+m_{D_{s1}1}}\right) A_{i'}^{B_c\to D_{s1}1} \sin\theta_s + \left(\frac{m_{B_c}+m_{D_{s1}}}{m_{B_c}+m_{D_{s1}2}}\right) A_{i'}^{B_c\to D_{s1}2} \cos\theta_s, \\
T_0^{B_c\to D_{s1}(2460)} &= \left(\frac{m_{B_c}+m_{D_{s1}1}}{m_{B_c}+m_{D_{s1}}}\right) T_0^{B_c\to D_{s1}1} \sin\theta_s + \left(\frac{m_{B_c}+m_{D_{s1}2}}{m_{B_c}+m_{D_{s1}}}\right) T_0^{B_c\to D_{s1}2} \cos\theta_s, \\
T_{j'}^{B_c\to D_{s1}(2460)} &= T_{j'}^{B_c\to D_{s1}1} \sin\theta_s + T_{j'}^{B_c\to D_{s1}2} \cos\theta_s.
\end{aligned} \tag{29}$$

where $i' = V, 1, 2$ and $j' = V, 1$. Note that, the $A_i^{B_c\to D_{s1}(2536)}$ and $T_j^{B_c\to D_{s1}(2536)}$ form factors are obtained from the above equations by replacing $\sin\theta_s \to \cos\theta_s$ and $\cos\theta_s \to -\sin\theta_s$.

4. Decay widths

Now, we present the dilepton invariant mass distribution for the $B_c \to D_{s1}\nu\bar{\nu}$ and $B_c \to D_{s1}l\bar{l}$ decays. Using the parameterization of the $B_c \to D_{s1}$ transition in terms of form factors and also equation (3), the dilepton invariant mass distribution of the $B_c \to D_{s1}\nu\bar{\nu}$ decay can be written as [43]

$$\frac{d\Gamma\left(B_c \to D_{s1}\nu\bar{\nu}\right)}{dq^2} = \frac{3G_F^2 m_{B_c}^3 |V_{tb}V_{ts}^*|^2 \alpha^2 |D\left(x_t\right)|^2}{2^8 \pi^5 \sin^4\theta_W} \phi_{D_{s1}}^{\frac{1}{2}} \left[s\alpha_1 + \frac{\phi_{D_{s1}}}{3}\beta_1\right], \tag{30}$$

where $s = q^2/m_{B_c}^2$, $x_t = m_t^2/m_W^2$. The parameters $D(x_t)$, $\phi_{D_{s1}}$, α_1 and β_1 are defined by

$$D(x_t) = \frac{x_t}{8}\left(\frac{2+x_t}{x_t-1} + \frac{3x_t-6}{(x_t-1)^2}\ln x_t\right),$$

$$\phi_{D_{s1}} = \left(1-r_{D_{s1}}\right)^2 - 2s\left(1+r_{D_{s1}}\right) + s^2,$$

$$\alpha_1 = \left(1-\sqrt{r_{D_{s1}}}\right)^2 \left|A_0^{B_c\to D_{s1}}\right|^2 + \frac{\phi_{D_{s1}}}{\left(1+\sqrt{r_{D_{s1}}}\right)^2}\left|A_V^{B_c\to D_{s1}}\right|^2,$$

$$\beta_1 = \frac{\left(1-\sqrt{r_{D_{s1}}}\right)^2}{4r_{D_{s1}}}\left|A_0^{B_c\to D_{s1}}\right|^2 - \frac{s}{\left(1+\sqrt{r_{D_{s1}}}\right)^2}\left|A_V^{B_c\to D_{s1}}\right|^2 + \frac{\phi_{D_{s1}}\left|A_1^{B_c\to D_{s1}}\right|^2}{4r_{D_{s1}}\left(1+\sqrt{r_{D_{s1}}}\right)^2} + \frac{1}{2}\left(\frac{1-s}{r_{D_{s1}}}-1\right)\frac{1-\sqrt{r_{D_{s1}}}}{1+\sqrt{r_{D_{s1}}}}\mathrm{Re}\left(A_0^{B_c\to D_{s1}}A_1^{*B_c\to D_{s1}}\right), \tag{31}$$

where $r_{D_{s1}} = m_{D_{s1}}^2/m_{B_c}^2$. The differential decay rates for $B_c \to D_{s1}l\bar{l}$ are found to be [44, 45]

$$\frac{d\Gamma\left(B_c^+ \to D_{s1}l^+l^-\right)}{dq^2} = \frac{G_F^2 m_{B_c}^3 |V_{tb}V_{ts}^*|^2 \alpha^2}{2^9\pi^5} v\phi_{D_{s1}}^{\frac{1}{2}}\left[\left(1+\frac{2m_l^2}{q^2}\right)\left(s\alpha_3 + \frac{\phi_{D_{s1}}}{3}\beta_3\right) + 4t\delta\right], \tag{32}$$

where $t = m_l^2/m_{B_c}^2$, $v = \sqrt{1-4m_l^2/q^2}$ and the expressions of α_3, β_3 and δ are given as follows:

$$\alpha_3 = \left(1-\sqrt{r_{D_{s1}}}\right)^2\left[\left|C_9^{\mathrm{eff}}A_0^{B_c\to D_{s1}} - \frac{2\hat{m}_b C_7^{\mathrm{eff}}\left(1+\sqrt{r_{D_{s1}}}\right)T_0^{B_c\to D_{s1}}}{s}\right|^2 + \left|C_{10}A_0^{B_c\to D_{s1}}\right|^2\right] + \frac{\phi_{D_{s1}}}{\left(1+\sqrt{r_{D_{s1}}}\right)^2}\left[\left|C_9^{\mathrm{eff}}A_V^{B_c\to D_{s1}} - \frac{2\hat{m}_b C_7^{\mathrm{eff}}\left(1+\sqrt{r_{D_{s1}}}\right)T_V^{B_c\to D_{s1}}}{s}\right|^2 + \left|C_{10}A_V^{B_c\to D_{s1}}\right|^2\right], \tag{33}$$

$$\beta_3 = \frac{\left(1-\sqrt{r_{D_{s1}}}\right)^2}{4r_{D_{s1}}}\left[\left|C_9^{\mathrm{eff}}A_0^{B_c\to D_{s1}} - \frac{2\hat{m}_b C_7^{\mathrm{eff}}\left(1+\sqrt{r_{D_{s1}}}\right)T_0^{B_c\to D_{s1}}}{s}\right|^2 + \left|C_{10}A_0^{B_c\to D_{s1}}\right|^2\right] - \frac{s}{\left(1+\sqrt{r_{D_{s1}}}\right)^2}\left[\left|C_9^{\mathrm{eff}}A_V^{B_c\to D_{s1}} - \frac{2\hat{m}_b C_7^{\mathrm{eff}}\left(1+\sqrt{r_{D_{s1}}}\right)T_V^{B_c\to D_{s1}}}{s}\right|^2 + \left|C_{10}A_V^{B_c\to D_{s1}}\right|^2\right] + \frac{\phi_{D_{s1}}}{4r_{D_{s1}}\left(1+\sqrt{r_{D_{s1}}}\right)^2}\left[\left|C_9^{\mathrm{eff}}A_1^{B_c\to D_{s1}} - \frac{2\hat{m}_b C_7^{\mathrm{eff}}\left(1+\sqrt{r_{D_{s1}}}\right)T_1^{B_c\to D_{s1}}}{s}\right|^2 + \left|C_{10}A_1^{B_c\to D_{s1}}\right|^2\right]$$

$$+\frac{1}{2}\left(\frac{1-s}{r_{D_{s1}}}-1\right)\frac{1-\sqrt{r_{D_{s1}}}}{1+\sqrt{r_{D_{s1}}}}\,\mathrm{Re}\left\{\left[C_9^{\mathrm{eff}}A_0^{B_c\to D_{s1}}-\frac{2\hat{m}_bC_7^{\mathrm{eff}}\left(1+\sqrt{r_{D_{s1}}}\right)T_0^{B_c\to D_{s1}}}{s}\right]\right.$$
$$\left.\times\left[C_9^{\mathrm{eff}}A_1^{B_c\to D_{s1}}-\frac{2\hat{m}_bC_7^{\mathrm{eff}}\left(1+\sqrt{r_{D_{s1}}}\right)T_1^{B_c\to D_{s1}}}{s}\right]+|C_{10}|^2\,\mathrm{Re}\left(A_0^{B_c\to D_{s1}}A_1^{*B_c\to D_{s1}}\right)\right\} \tag{34}$$

and

$$\delta=\frac{|C_{10}|^2}{2\left(1+\sqrt{r_{D_{s1}}}\right)^2}\left\{-2\phi_{D_{s1}}\left|A_V^{B_c\to D_{s1}}\right|^2-3\left(1-r_{D_{s1}}\right)^2\left|A_0^{B_c\to D_{s1}}\right|^2\right.$$
$$+\frac{\phi_{D_{s1}}}{4r_{D_{s1}}}\left[2\left(1+r_{D_{s1}}\right)-s\right]\left|A_1^{B_c\to D_{s1}}\right|^2+\frac{\phi_{D_{s1}}s}{4r_{D_{s1}}}\left|A_2^{B_c\to D_{s1}}\right|^2$$
$$\left.+\frac{\phi_{D_{s1}}\left(1-r_{D_{s1}}\right)}{2r_{D_{s1}}}\mathrm{Re}\left(A_0^{B_c\to D_{s1}}A_1^{*B_c\to D_{s1}}+A_0^{B_c\to D_{s1}}A_2^{*B_c\to D_{s1}}+A_1^{B_c\to D_{s1}}A_2^{*B_c\to D_{s1}}\right)\right\}, \tag{35}$$

where $\hat{m}_b=m_b/m_{B_c}$.

5. Numerical analysis

In this section, we present our numerical analysis of the form factors A_i ($i=V,0,1,2$) and T_j, ($j=V,0,1$). From the sum rule expressions of the form factors, it is clear that the main input parameters entering the expressions are gluon condensates, elements of the CKM matrix V_{tb} and V_{ts}, leptonic decay constants f_{B_c}, $f_{D_{s1}}$, $f_{D_{s1}1}$ and $f_{D_{s1}2}$, Borel parameters M_1^2 and M_2^2 as well as the continuum thresholds s_0 and s_0'. We choose the values of the condensates (at a fixed renormalization scale of about 1 GeV), leptonic decay constants , CKM matrix elements, quark and meson masses as follows: $\langle\frac{\alpha_s}{\pi}G^2\rangle=0.012\ \mathrm{GeV}^4$ [46], $|V_{tb}|=0.77^{+0.18}_{-0.24}$, $|V_{ts}|=(40.6\pm2.7)\times10^{-3}$ [47], $C_7^{\mathrm{eff}}=-0.313$, $C_9^{\mathrm{eff}}=4.344$, $C_{10}=-4.669$ [48, 49], $f_{D_{s1}}=(225\pm25)$ MeV, $f_{D_{s1}1}=(240\pm25)$ MeV, $f_{D_{s1}2}=(63\pm7)$ MeV [36, 50], $f_{B_c}=(350\pm25)$ MeV [51–53], $m_s(1\ \mathrm{GeV})=\left(104^{+26}_{-34}\right)$ MeV, $m_c=\left(1.27^{+0.07}_{-0.11}\right)$ GeV, $m_b=(4.7\pm0.07)$ GeV, $m_{D_{s1}(2460)}=(2459.6\pm0.6)$ MeV, $m_{D_{s1}(2536)}=(2535.35\pm0.34\pm0.5)$ MeV and $m_{B_c}=(6.276\pm0.004)$ GeV [54].

The sum rules for the form factors also contain four auxiliary parameters: Borel mass squares M_1^2 and M_2^2 and continuum thresholds s_0 and s_0'. These are not physical quantities, so the form factors as physical quantities should be independent of them. The parameters s_0 and s_0', which are the continuum thresholds of B_c and D_{s1} mesons, respectively, are determined from the condition that guarantees the sum rules to practically be stable in the allowed regions for M_1^2 and M_2^2. The values of the continuum thresholds calculated from the two-point QCD sum rules are taken to be $s_0=(45–50)\ \mathrm{GeV}^2$ and $s_0'=(6–8)\ \mathrm{GeV}^2$ [46, 55, 56]. The working regions for M_1^2 and M_2^2 are determined requiring that not only the contributions of the higher states and continuum are small, but the contributions of the operators with higher dimensions are also small. Both conditions are satisfied in the regions $10\ \mathrm{GeV}^2\leqslant M_1^2\leqslant25\ \mathrm{GeV}^2$ and $4\ \mathrm{GeV}^2\leqslant M_2^2\leqslant10\ \mathrm{GeV}^2$. First, we would like to consider the D_{s1} meson as the pure $|c\bar{s}\rangle$ state. The values of the form factors at $q^2=0$ are presented in table 2. The sum rules for the form factors are truncated at about $10\ \mathrm{GeV}^2$, so to extend our results to the full physical region, $0\leqslant q^2\leqslant\left(m_{B_c}-m_{D_{s1}}\right)^2\ \mathrm{GeV}^2$, we look for a parameterization of the form factors in such a way that in the region $0\leqslant q^2\leqslant10\ \mathrm{GeV}^2$, this parameterization coincides with the sum rule predictions. Our numerical calculations show that the sufficient parameterization of

Table 2. The value of the form factors of the $B_c \to D_{s1}(2460)$ and $B_c \to D_{s1}(2536)$ transitions at $q^2 = 0$, $M_1^2 = 15$ GeV^2 and $M_2^2 = 8$ GeV^2, when the D_{s1} mesons are considered as the pure $|c\bar{s}\rangle$ state.

$A_V^{B_c\to D_{s1}(2460)}(0)$	-0.23 ± 0.07	$A_V^{B_c\to D_{s1}(2536)}(0)$	-0.22 ± 0.06
$A_0^{B_c\to D_{s1}(2460)}(0)$	0.09 ± 0.02	$A_0^{B_c\to D_{s1}(2536)}(0)$	0.07 ± 0.02
$A_1^{B_c\to D_{s1}(2460)}(0)$	0.16 ± 0.05	$A_1^{B_c\to D_{s1}(2536)}(0)$	0.17 ± 0.05
$A_2^{B_c\to D_{s1}(2460)}(0)$	-0.26 ± 0.08	$A_2^{B_c\to D_{s1}(2536)}(0)$	-0.28 ± 0.09
$T_V^{B_c\to D_{s1}(2460)}(02)$	0.12 ± 0.03	$T_V^{B_c\to D_{s1}(2536)}(0)$	0.14 ± 0.04
$T_0^{B_c\to D_{s1}(2460)}(0)$	0.11 ± 0.03	$T_0^{B_c\to D_{s1}(2536)}(0)$	0.14 ± 0.04
$T_1^{B_c\to D_{s1}(2460)}(0)$	-0.14 ± 0.04	$T_1^{B_c\to D_{s1}(2536)}(0)$	-0.16 ± 0.05

Table 3. Parameters appearing in the fit function for the form factors of the $B_c \to D_{s1}(2460)$ and $B_c \to D_{s1}(2536)$ transitions at $M_1^2 = 15$ GeV^2 and $M_2^2 = 8$ GeV^2.

	a	b	$m_{\rm fit}$		a	b	$m_{\rm fit}$
$A_V^{B_c\to D_{s1}(2460)}(q^2)$	-0.13	-0.10	5.30	$A_V^{B_c\to D_{s1}(2536)}(q^2)$	-0.12	-0.10	5.22
$A_0^{B_c\to D_{s1}(2460)}(q^2)$	0.05	0.04	5.98	$A_0^{B_c\to D_{s1}(2536)}(q^2)$	0.04	0.03	5.99
$A_1^{B_c\to D_{s1}(2460)}(q^2)$	0.09	0.07	5.95	$A_1^{B_c\to D_{s1}(2536)}(q^2)$	0.09	0.08	5.98
$A_2^{B_c\to D_{s1}(2460)}(q^2)$	-0.16	-0.10	5.15	$A_2^{B_c\to D_{s1}(2536)}(q^2)$	-0.17	-0.11	5.17
$T_V^{B_c\to D_{s1}(2460)}(q^2)$	0.08	0.04	5.22	$T_V^{B_c\to D_{s1}(2536)}(q^2)$	0.09	0.05	5.05
$T_0^{B_c\to D_{s1}(2460)}(q^2)$	0.10	0.01	5.85	$T_0^{B_c\to D_{s1}(2536)}(q^2)$	0.11	0.03	5.96
$T_1^{B_c\to D_{s1}(2460)}(q^2)$	-0.09	-0.05	5.22	$T_1^{B_c\to D_{s1}(2536)}(q^2)$	-0.08	-0.07	5.14

Table 4. The branching ratios of the semileptonic $B_c \to D_{s1}(2460)l^+l^-/\nu\bar{\nu}$ and $B_c \to D_{s1}(2536)l^+l^-/\nu\bar{\nu}$ decays with SD effects.

MODS	BR	MODS	BR
$B_c \to D_{s1}(2460)\nu\bar{\nu}$	$(3.26 \pm 1.10) \times 10^{-7}$	$B_c \to D_{s1}(2536)\nu\bar{\nu}$	$(2.76 \pm 0.88) \times 10^{-7}$
$B_c \to D_{s1}(2460)e^+e^-$	$(5.40 \pm 1.70) \times 10^{-6}$	$B_c \to D_{s1}(2536)e^+e^-$	$(2.91 \pm 0.93) \times 10^{-6}$
$B_c \to D_{s1}(2460)\mu^+\mu^-$	$(2.27 \pm 0.95) \times 10^{-6}$	$B_c \to D_{s1}(2536)\mu^+\mu^-$	$(1.96 \pm 0.63) \times 10^{-6}$
$B_c \to D_{s1}(2460)\tau^+\tau^-$	$(1.42 \pm 0.45) \times 10^{-8}$	$B_c \to D_{s1}(2536)\tau^+\tau^-$	$(0.68 \pm 0.21) \times 10^{-8}$

the form factors with respect to q^2 is as follows:

$$f_i(q^2) = \frac{a}{\left(1 - \frac{q^2}{m_{\rm fit}^2}\right)} + \frac{b}{\left(1 - \frac{q^2}{m_{\rm fit}^2}\right)^2}. \tag{36}$$

The values of the parameters a, b and $m_{\rm fit}$ are given in table 3. To calculate the branching ratios of the $B_c \to D_{s1}(2460[2536])l^+l^-/\nu\bar{\nu}$ decays, we integrate equations (30) and (32) over q^2 in the whole physical region and use the total mean lifetime $\tau_{B_c} = (0.46 \pm 0.07)\,ps$ [54]. Our numerical analysis shows that the contribution of the non-perturbative part (the gluon condensate diagrams) is about 12% of the total and the main contribution comes from the perturbative part of the form factors. The values for the branching ratio of these decays are obtained as presented in table 4, when only the short distance (SD) effects are considered. It should be noted that the long distance (LD) effects for the charged lepton modes are not included in the values of table 4. With the LD effects, we introduce some cuts close to $q^2 = 0$

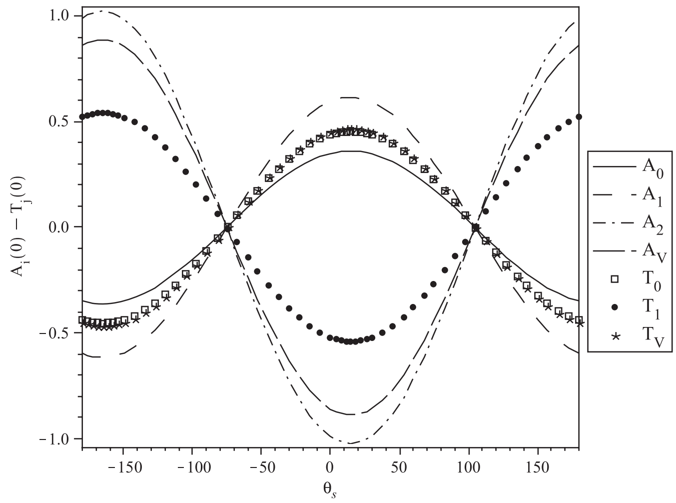

Figure 3. The dependence of the form factors on θ_s at $q^2 = 0$ for $B_c \to D_{s1}(2460)$ transition.

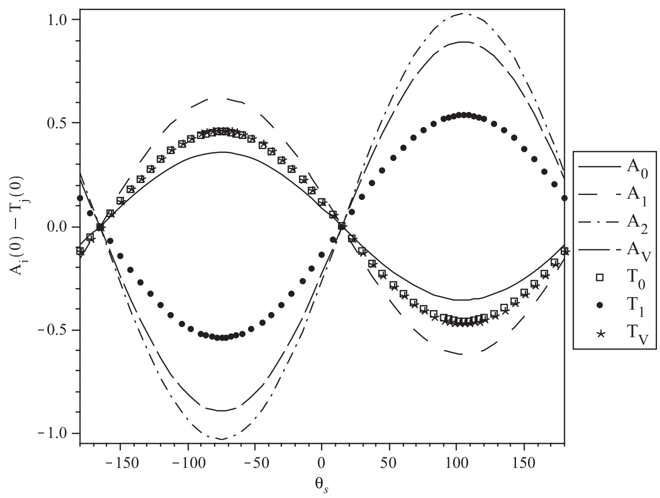

Figure 4. The dependence of the form factors on θ_s at $q^2 = 0$ for $B_c \to D_{s1}(2536)$ transition.

and around the resonances of J/ψ and ψ' and study the three regions as follows:

$$\begin{aligned} I&: \quad \sqrt{q^2_{\min}} \leqslant \sqrt{q^2} \leqslant M_{J/\psi} - 0.20, \\ II&: \quad M_{J/\psi} + 0.04 \leqslant \sqrt{q^2} \leqslant M_{\psi'} - 0.10, \\ III&: \quad M_{\psi'} + 0.02 \leqslant \sqrt{q^2} \leqslant m_{B_c} - m_{D_{s1}}, \end{aligned} \tag{37}$$

where $\sqrt{q^2_{\min}} = 2m_l$. In table 5, we present the branching ratios in terms of the regions shown in equation (37). The errors are estimated by the variation of the Borel parameters M_1^2 and M_2^2, the variation of the continuum thresholds s_0 and s'_0, and the variation of b and c quark masses and leptonic decay constants f_{B_c} and $f_{D_{s1}}$.

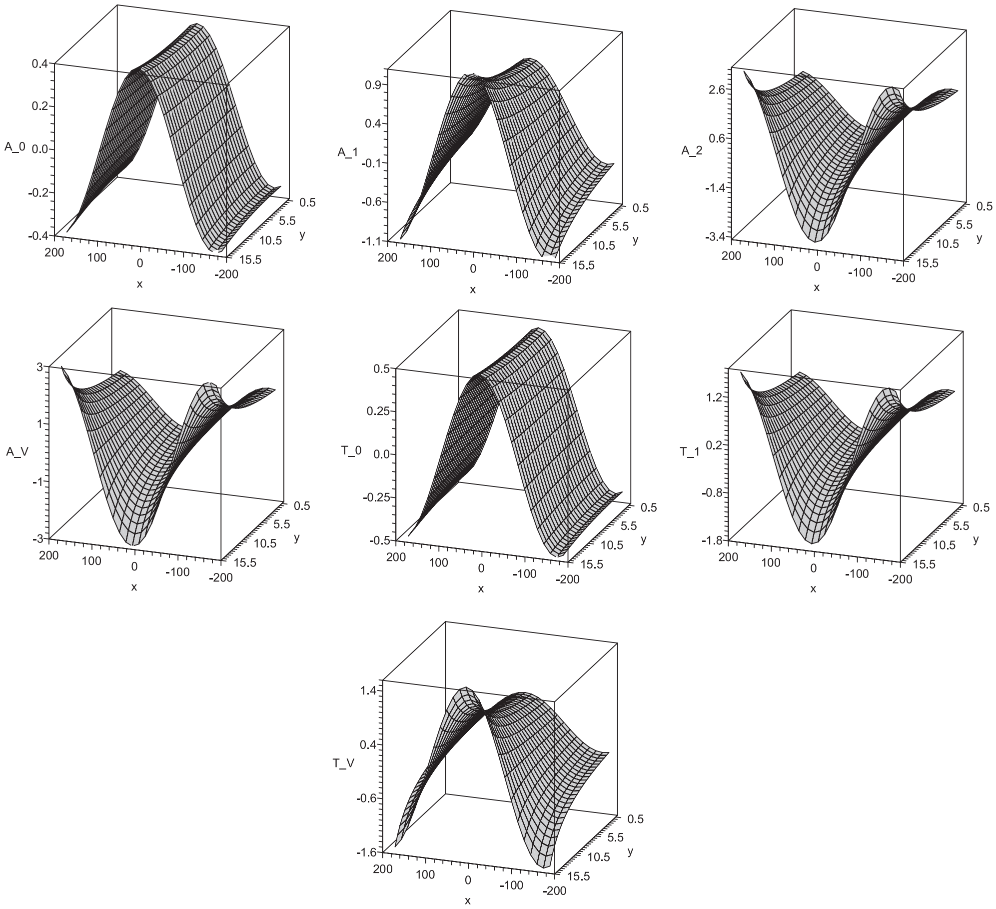

Figure 5. The dependence of the transition form factors on q^2 and θ_s for $B_c \to D_{s1}(2460)$ transition. In these figures, $x = \theta_s$ and $y = q^2$.

Table 5. The branching ratios of the semileptonic $B_c \to D_{s1}(2460)l^+l^-/\nu\bar{\nu}$ and $B_c \to D_{s1}(2536)l^+l^-/\nu\bar{\nu}$ decays including LD effects.

MODS	*I*	*II*	*III*
$BR(B_c \to D_{s1}(2460)e^+e^-)$	$(4.40 \pm 1.35) \times 10^{-6}$	$(1.62 \pm 0.52) \times 10^{-7}$	$(1.01 \pm 0.35) \times 10^{-7}$
$BR(B_c \to D_{s1}(2536)e^+e^-)$	$(2.65 \pm 0.82) \times 10^{-6}$	$(0.90 \pm 0.28) \times 10^{-7}$	$(0.81 \pm 0.25) \times 10^{-7}$
$BR(B_c \to D_{s1}(2460)\mu^+\mu^-)$	$(1.76 \pm 0.58) \times 10^{-6}$	$(1.61 \pm 0.53) \times 10^{-7}$	$(1.01 \pm 0.35) \times 10^{-7}$
$BR(B_c \to D_{s1}(2536)\mu^+\mu^-)$	$(1.00 \pm 0.31) \times 10^{-6}$	$(0.96 \pm 0.29) \times 10^{-7}$	$(0.68 \pm 0.21) \times 10^{-7}$
$BR(B_c \to D_{s1}(2460)\tau^+\tau^-)$	Undefined	$(5.20 \pm 1.61) \times 10^{-9}$	$(8.12 \pm 2.75) \times 10^{-9}$
$BR(B_c \to D_{s1}(2536)\tau^+\tau^-)$	Undefined	$(4.16 \pm 1.29) \times 10^{-9}$	$(6.56 \pm 2.16) \times 10^{-9}$

Now, we would like to analyze the form factors obtained when we considered the D_{s1} meson as a mixture of two $|^3P_1\rangle$ and $|^1P_1\rangle$ states (see equation (29)). The transition form factors of $B_c \to D_{s1}(2460[2536])l^+l^-/\nu\bar{\nu}$ at $q^2 = 0$ in the interval $-180^\circ \leqslant \theta_s \leqslant 180^\circ$ are shown in figures 3 and 4. From these figures, we see that all form factors have the following common

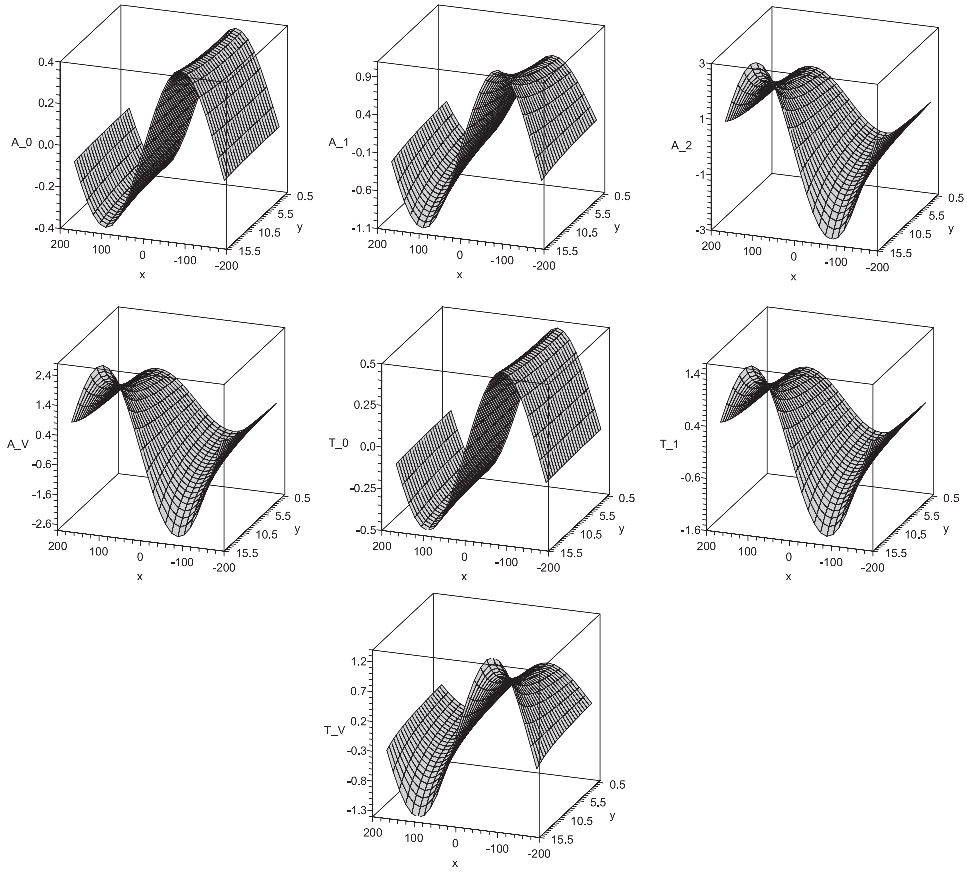

Figure 6. The dependence of the transition form factors on q^2 and θ_s for $B_c \to D_{s1}(2536)$ transition. In these figures, $x = \theta_s$ and $y = q^2$.

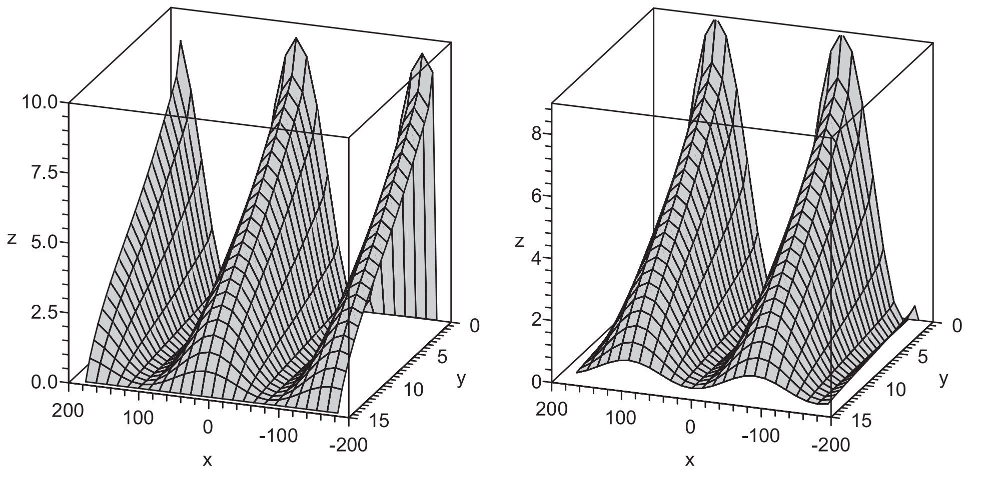

Figure 7. The decay width for $B_c \to D_{s1}\mu^+\mu^-$ with respect to θ_s and q^2. The left figure shows decay width of $B_c \to D_{s1}(2460)\mu^+\mu^-$ and right figure belongs to $B_c \to D_{s1}(2536)\mu^+\mu^-$. In these figures, $x = \theta_s$, $y = q^2$ and $z = \Gamma(B_c \to D_{s1}\mu^+\mu^-) \times 10^{-20}$.

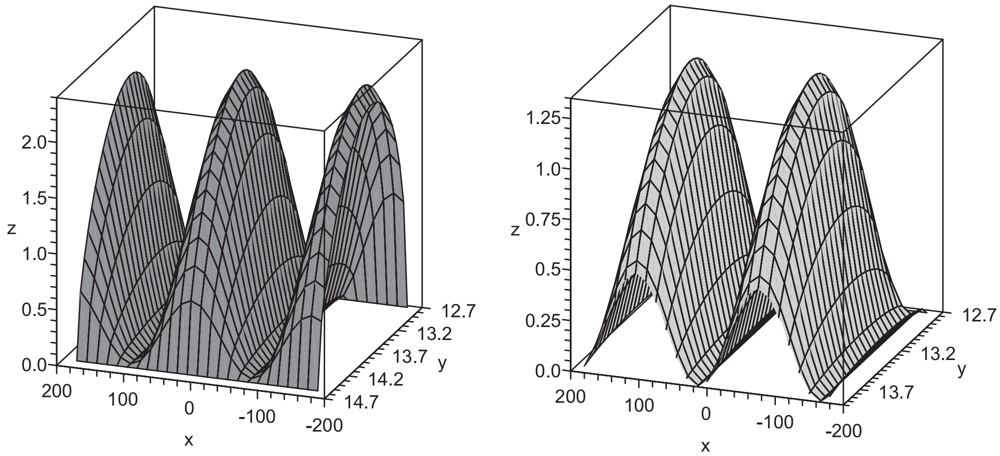

Figure 8. The decay width for $B_c \to D_{s1}\tau^+\tau^-$ with respect to θ_s and q^2. The left figure shows decay width of $B_c \to D_{s1}(2460)\tau^+\tau^-$ and right figure belongs to $B_c \to D_{s1}(2536)\tau^+\tau^-$. In these figures, $x = \theta_s$, $y = q^2$ and $z = \Gamma(B_c \to D_{s1}\tau^+\tau^-) \times 10^{-21}$.

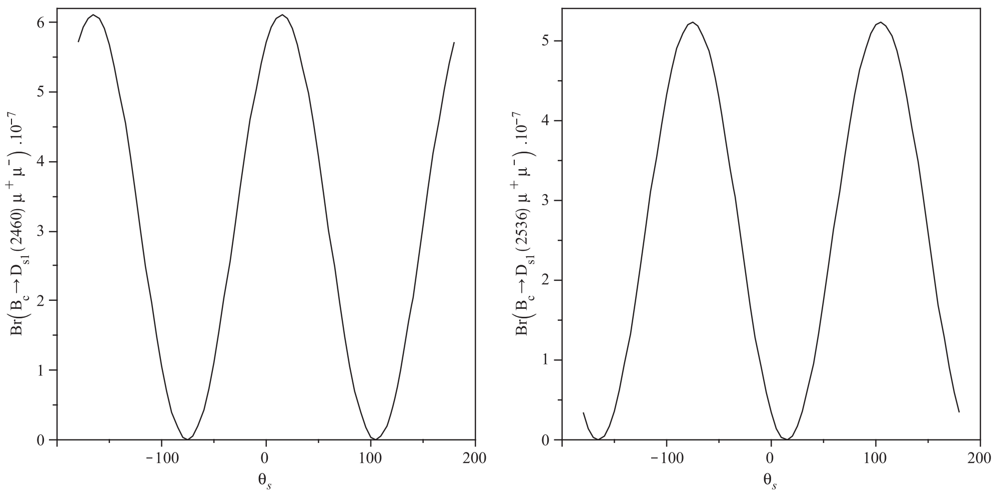

Figure 9. The branching ratios of $B_c \to D_{s1}\mu^+\mu^-$ with respect to θ_s.

behaviors: (1) they have extrema at the same mixing angles and (2) they come across at two points. The dependence of the form factors $B_c \to D_{s1}(2460)$ and $B_c \to D_{s1}(2536)$ transitions on the mixing angle, θ_s, and the transferred momentum square, q^2, are plotted in figures 5 and 6 in the regions $0 \leqslant q^2 \leqslant (m_{B_c} - m_{D_{s1}})^2$ GeV^2 and $-180^\circ \leqslant \theta_s \leqslant 180^\circ$ for q^2 and mixing angle, respectively. Using equations (30) and (32), we analyze the decay widths and the branching ratios related to considered decays. For this aim, we denote the variations of the decay widths with respect to q^2 and θ_s in the regions $4m_l^2 \leqslant q^2 \leqslant (m_{B_c} - m_{D_{s1}})^2$ GeV^2 $(l = \tau, \mu, e)$ and $-180^\circ \leqslant \theta_s \leqslant 180^\circ$ and branching ratios only in terms of mixing angle θ_s in figures 7–10. The results for electron and muon are approximately the same, so we consider $l = \mu, \tau$. Figures 11 and 12 depict those quantities, but for $B_c \to D_{s1}(2460[2536])\nu\bar{\nu}$ decays. Figures 9 and 10 depict a regular variation of the branching ratios for l^+l^- case with respect to

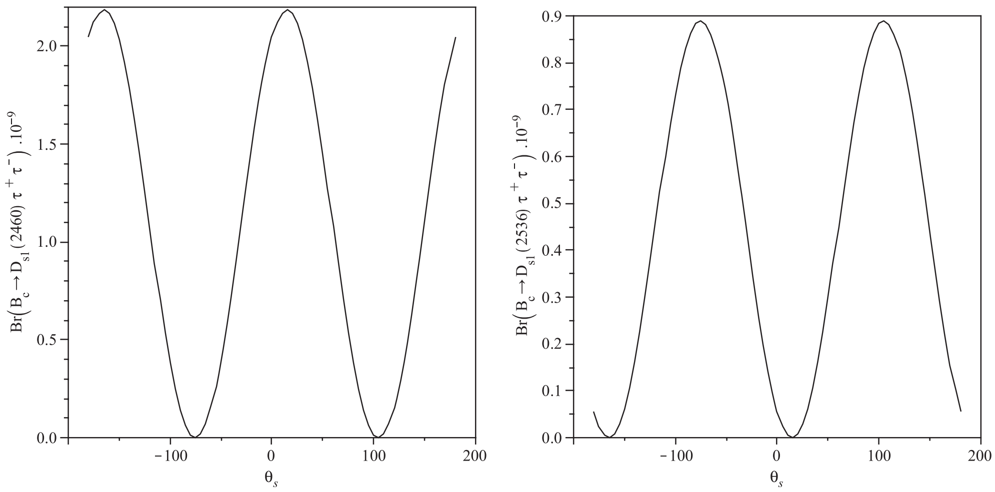

Figure 10. The branching ratios of $B_c \to D_{s1}\tau^+\tau^-$ with respect to θ_s.

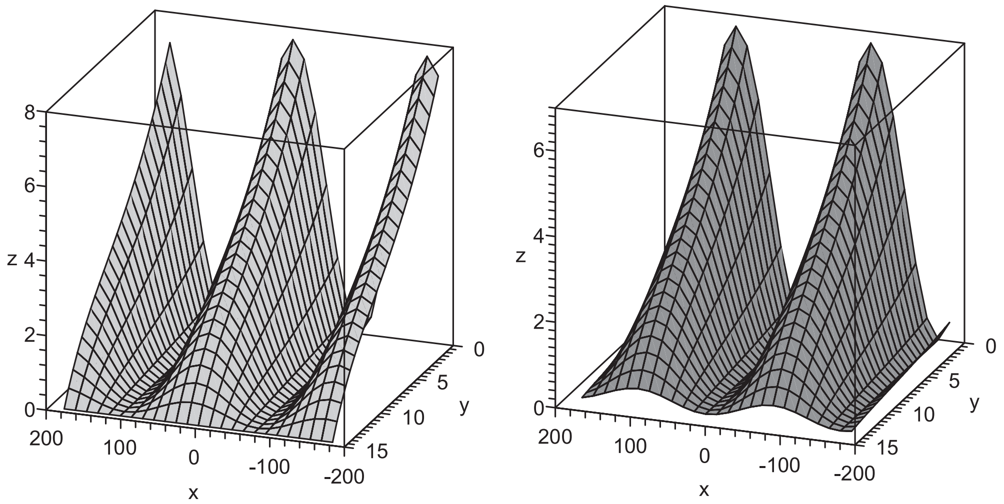

Figure 11. Same as figure 7, but for $B_c \to D_{s1}\nu\bar{\nu}$ and $z = \Gamma(B_c \to D_{s1}\nu\bar{\nu}) \times 10^{-19}$.

the mixing angle, while we see an irregular variation of the branching ratios of the $B_c \to D_{s1}(2460, 2536)\nu\bar{\nu}$ transitions with respect to θ_s.

In summary, we analyzed the semileptonic $B_c \to D_{s1}(2460[2536])l^+l^-$, $l = e, \mu, \tau$ and $B_c \to D_{s1}(2460[2536])\nu\bar{\nu}$ decays in the framework of the three-point QCD sum rules. First, we assumed the $D_{s1}(2460)$ and $D_{s1}(2536)$ axial vector mesons as the pure $|c\bar{s}\rangle$ states. In this case, the related form factors were computed. The branching ratios of these decays were also estimated with both the short distance and long distances effects, for the charged lepton modes. Second, $D_{s1}(2460[2536])$ mesons were considered as combinations of two states $|^3P_1\rangle \equiv |D_{s1}1\rangle$ and $|^1P_1\rangle \equiv |D_{s1}2\rangle$ with different masses and decay constants. We evaluated the transitions form factors and the decay widths of these decays with respect to the mixing angle θ_s and the transferred momentum square q^2. The dependence of the branching

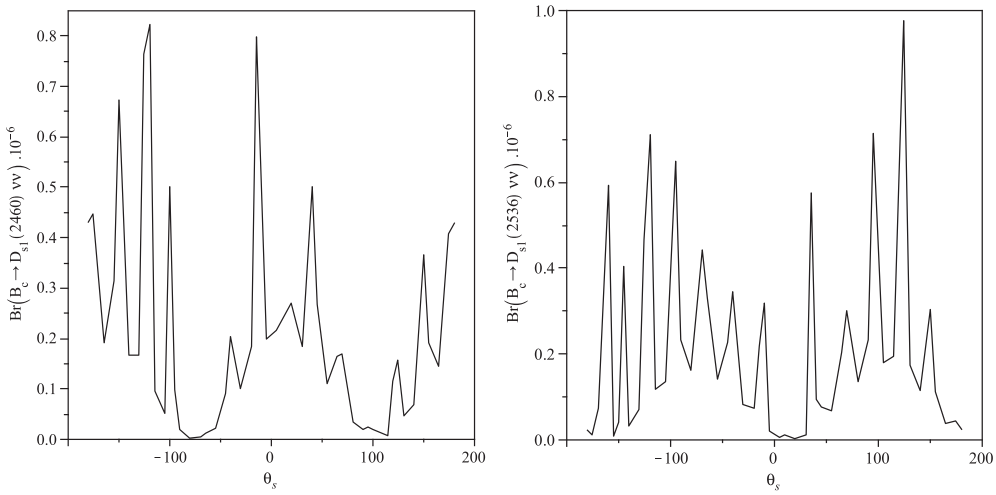

Figure 12. Same as figure 9, but for $B_c \to D_{s1}\nu\bar{\nu}$.

ratios on θ_s was also presented. Detection of these channels and their comparison with the phenomenological models like QCD sum rules could give useful information about the structure of the D_{s1} meson and the mixing angle θ_s.

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Appendix

In this appendix, the explicit expressions of the coefficients of the gluon condensate entering the sum rules of the form factors $A_i^{\prime B_c\to D_{s1}}$ $(i = V, 0, 1, 2)$ and $T_j^{\prime B_c\to D_{s1}}$, $(j = V, 0, 1)$ are given

$$
\begin{aligned}
C_V^{V-A} = {} & -10\hat{I}_1(3,2,2)m_b^3m_c^2 + 10\hat{I}_1(3,2,2)m_b^2m_c^3 + 10\hat{I}_2(3,2,2)m_b^2m_c^3 \\
& + 10\hat{I}_0(3,2,2)m_b^2m_c^3 + 10\hat{I}_1(3,2,2)m_bm_c^4 + 10\hat{I}_0^{[0,1]}(3,2,2)m_b^2m_c \\
& - 30\hat{I}_1(3,2,1)m_b^2m_c + 60\hat{I}_1(1,4,1)m_b^2m_c + 60\hat{I}_2(1,4,1)m_b^2m_c \\
& - 20\hat{I}_2(3,2,1)m_b^2m_c + 10\hat{I}_2^{[0,1]}(3,2,2)m_b^2m_c - 20\hat{I}_0(3,2,1)m_b^2m_c \\
& + 60\hat{I}_0(1,4,1)m_b^2m_c + 10\hat{I}_1^{[0,1]}(3,2,2)m_b^2m_c + 20\hat{I}_1(2,2,2)m_bm_c^2 \\
& + 10\hat{I}_2(3,2,1)m_bm_c^2 + 10\hat{I}_1(3,2,1)m_bm_c^2 + 40\hat{I}_2(2,3,1)m_bm_c^2 \\
& - 10\hat{I}_0(3,2,1)m_bm_c^2 + 20\hat{I}_1(2,3,1)m_bm_c^2 - 20\hat{I}_1^{[0,1]}(3,2,2)m_bm_c^2 \\
& + 30\hat{I}_1(4,1,1)m_b{m_c}^2 - 10\hat{I}_2(3,2,2)m_c^5 - 10\hat{I}_1(3,2,2)m_c^5 \\
& - 10\hat{I}_0(3,2,2)m_c^5 + 20\hat{I}_1(3,2,1)m_b^3 + 10\hat{I}_1(2,2,2)m_b^3 - 20\hat{I}_1(2,3,1)m_b^3 \\
& - 60\hat{I}_1(1,4,1)m_b^3 - 10\hat{I}_1^{[0,1]}(3,2,2)m_b^3 - 30\hat{I}_2(4,1,1)m_c^3 + 20\hat{I}_2^{[0,1]}(3,2,2)m_c^3
\end{aligned}
$$

$$
\begin{aligned}
&+10\hat{I}_0(3,2,1)m_c^3-10\hat{I}_2(3,1,2)m_c^3-20\hat{I}_0(2,2,2)m_c^3-20\hat{I}_2(2,2,2)m_c^3\\
&-20\hat{I}_1(2,2,2)m_c^3-30\hat{I}_0(4,1,1)m_c^3-30\hat{I}_1(4,1,1)m_c^3+20\hat{I}_1^{[0,1]}(3,2,2)m_c^3\\
&-10\hat{I}_0(3,1,2)m_c^3+20\hat{I}_0^{[0,1]}(3,2,2)m_c^3-50\hat{I}_1(2,2,1)m_b+20\hat{I}_1^{[0,1]}(2,3,1)m_b\\
&-20\hat{I}_1^{[0,1]}(3,2,1)m_b+20\hat{I}_1(1,2,2)m_b+60\hat{I}_0(1,3,1)m_b-20\hat{I}_2(2,2,1)m_b\\
&-20\hat{I}_1^{[0,1]}(3,1,2)m_b-20\hat{I}_0(2,2,1)m_b+30\hat{I}_1(2,1,2)m_b+100\hat{I}_2(1,3,1)m_b\\
&+10\hat{I}_1^{[0,2]}(3,2,2)m_b-20\hat{I}_1^{[0,1]}(2,2,2)m_b+40\hat{I}_0^{[0,1]}(2,3,1)m_b+20\hat{I}_1(1,3,1)m_b\\
&+30\hat{I}_0(2,2,1)m_c+30\hat{I}_2^{[0,1]}(3,1,2)m_c+20\hat{I}_2^{[0,1]}(3,2,1)m_c+10\hat{I}_0^{[0,1]}(3,2,1)m_c\\
&+20\hat{I}_1^{[0,1]}(3,2,1)m_c-10\hat{I}_0^{[0,2]}(3,2,2)m_c+20\hat{I}_1^{[0,1]}(3,1,2)m_c+20\hat{I}_1^{[0,1]}(2,2,2)m_c\\
&+20\hat{I}_2(2,2,1)m_c-30\hat{I}_2(2,1,2)m_c+10\hat{I}_0(3,1,1)m_c+20\hat{I}_0^{[0,1]}(2,2,2)m_c\\
&-20\hat{I}_0(1,2,2)m_c-20\hat{I}_2(1,2,2)m_c+30\hat{I}_0^{[0,1]}(3,1,2)m_c-10\hat{I}_1(3,1,1)m_c\\
&+20\hat{I}_2^{[0,1]}(2,2,2)m_c-10\hat{I}_2(3,1,1)m_c-20\hat{I}_1(2,1,2)m_c-30\hat{I}_0(2,1,2)m_c\\
&-10\hat{I}_2^{[0,2]}(3,2,2)m_c+20\hat{I}_1(2,2,1)m_c-20\hat{I}_1(1,2,2)m_c-10\hat{I}_1^{[0,2]}(3,2,2)m_c
\end{aligned}
$$

$$
\begin{aligned}
C_0^{V-A}={}&-20\hat{I}_6(3,2,2)m_c^5-40\hat{I}_6(3,2,1)m_c^3-20\hat{I}_6(3,1,2)m_c^3+40\hat{I}_6^{[0,6]}(3,2,2)m_c^3\\
&-40\hat{I}_6(2,2,2)m_c^3-60\hat{I}_6(4,1,1)m_c^3+5\hat{I}_0(3,1,1)m_c^3-30\hat{I}_0^{[0,1]}(1,4,1)m_b^3\\
&+20\hat{I}_6(2,2,2)m_b^3+5\hat{I}_0(2,2,1)m_b^3-120\hat{I}_6(1,4,1)m_b^3+40\hat{I}_6(2,3,1)m_b^3\\
&+15\hat{I}_0^{[0,1]}(3,2,1)m_b^3-20\hat{I}_0(1,3,1)m_b^3+10\hat{I}_0^{[0,1]}(2,3,1)m_b^3-5\hat{I}_0^{[0,2]}(3,2,2)m_b^3\\
&+10\hat{I}_0^{[0,1]}(2,2,2)m_b^3-5\hat{I}_0(1,2,2)m_b^3-20\hat{I}_6^{[0,1]}(3,2,2)m_b^3+20\hat{I}_6^{[0,1]}(3,1,2)m_c\\
&+20\hat{I}_6(2,2,1)m_c+40\hat{I}_6^{[0,1]}(2,2,2)m_c-20\hat{I}_6^{[0,2]}(3,2,2)m_c+20\hat{I}_6(3,2,1)m_b^3\\
&+5\hat{I}_0^{[0,1]}(3,1,1)m_c+5\hat{I}_0(1,1,2)m_c+20\hat{I}_6(2,1,2)m_c+40\hat{I}_6(3,1,1)m_c\\
&+5\hat{I}_0(1,2,1)m_c-5\hat{I}_0(2,1,1)m_c-10\hat{I}_0^{[0,2]}(2,3,1)m_b-10\hat{I}_0^{[0,2]}(3,1,2)m_b\\
&-10\hat{I}_0^{[0,1]}(1,3,1)m_b-15\hat{I}_0(1,2,1)m_b-40\hat{I}_6(2,2,1)m_b+15\hat{I}_0^{[0,1]}(2,2,1)m_b\\
&+20\hat{I}_0^{[0,1]}(1,2,2)m_b-40\hat{I}_6(1,3,1)m_b-40\hat{I}_6(3,1,1)m_b+40\hat{I}_6^{[0,1]}(3,2,1)m_c\\
&-20\hat{I}_6^{[0,1]}(2,2,2)m_b+20\hat{I}_6^{[0,2]}(3,2,2)m_b-40\hat{I}_6^{[0,1]}(3,1,2)m_b-15\hat{I}_0(1,1,2)m_b\\
&-15\hat{I}_0^{[0,2]}(2,2,2)m_b+15\hat{I}_0(2,1,1)m_b-20\hat{I}_6(2,1,2)m_b+25\hat{I}_0^{[0,1]}(2,1,2)m_b\\
&+10\hat{I}_0^{[0,1]}(3,1,1)m_b-15\hat{I}_0^{[0,2]}(3,2,1)m_b-20\hat{I}_6(1,2,2)m_b-40\hat{I}_6^{[0,1]}(2,3,1)m_b\\
&-20\hat{I}_6^{[0,1]}(3,2,1)m_b-5\hat{I}_0(3,2,2)m_c^6m_b+10\hat{I}_0(2,2,2)m_c^3m_b^2+20\hat{I}_6(3,2,2)m_c^3m_b^2\\
&-10\hat{I}_0(2,3,1)m_c^4m_b+15\hat{I}_0^{[0,1]}(3,2,2)m_c^4m_b+20\hat{I}_6(3,2,2)m_c^4m_b\\
&-15\hat{I}_0(2,2,2)m_c^4m_b-5\hat{I}_0(3,2,1)m_c^4m_b-15\hat{I}_0(4,1,1)m_c^4m_b\\
&-5\hat{I}_0(3,2,2)m_c^3m_b^4+5\hat{I}_0(3,2,2)m_c^4m_b^3+5\hat{I}_0(3,2,2)m_c^5m_b^2-30\hat{I}_0(1,4,1)m_cm_b^4\\
&-5\hat{I}_0^{[0,1]}(3,2,2)m_cm_b^4+10\hat{I}_0(3,2,1)m_cm_b^4+10\hat{I}_0(2,3,1)m_c^2m_b^3\\
&-10\hat{I}_0(3,2,1)m_c^2m_b^3-20\hat{I}_6(3,2,2)m_c^2m_b^3+30\hat{I}_0(1,4,1)m_c^2m_b^3\\
&-10\hat{I}_0^{[0,1]}(3,2,2)m_c^3m_b^2+5\hat{I}_0(3,2,1)m_c^3m_b^2+15\hat{I}_0(4,1,1)m_c^3m_b^2\\
&+20\hat{I}_6(2,2,2)m_c^2m_b+10\hat{I}_0^{[0,1]}(3,1,2)m_c^2m_b-40\hat{I}_6^{[0,1]}(3,2,2){m_c}^2m_b\\
&-15\hat{I}_0^{[0,2]}(3,2,2)m_c^2m_b+10\hat{I}_0(1,3,1)m_c^2m_b+20\hat{I}_0^{[0,1]}(3,2,1)m_c^2m_b\\
&-20\hat{I}_0(1,2,2)m_c^2m_b-15\hat{I}_0(2,1,2)m_c^2m_b-40\hat{I}_6(2,3,1)m_c^2m_b\\
&+15\hat{I}_0^{[0,1]}(4,1,1)m_c^2m_b+30\hat{I}_0^{[0,1]}(2,2,2)m_c^2m_b+60\hat{I}_6(4,1,1)m_c^2m_b
\end{aligned}
$$

$$- 10\hat{I}_0(3,1,1)m_c^2m_b + 20\hat{I}_6(3,1,2)m_c^2m_b + 15\hat{I}_0(2,2,1)m_c^2m_b$$
$$+ 20\hat{I}_0^{[0,1]}(2,3,1)m_c^2m_b + 40\hat{I}_6(3,2,1)m_c^2m_b - 10\hat{I}_0^{[0,1]}(2,2,2)m_c{m_b}^2$$
$$- 20\hat{I}_6(3,2,1)m_cm_b^2 + 20\hat{I}_6^{[0,1]}(3,2,2)m_cm_b^2 + 15\hat{I}_0(2,1,2)m_cm_b^2$$
$$+ 5\hat{I}_0(3,1,1)m_cm_b^2 - 20\hat{I}_0^{[0,1]}(3,1,2)m_cm_b^2 - 20\hat{I}_6(2,2,2)m_cm_b^2$$
$$- 30\hat{I}_0(1,3,1)m_cm_b^2 + 120\hat{I}_6(1,4,1)m_cm_b^2 + 10\hat{I}_0(1,2,2)m_cm_b^2$$
$$- 5\hat{I}_0^{[0,1]}(3,2,1)m_cm_b^2 - 10\hat{I}_0(2,2,1)m_cm_b^2 + 5\hat{I}_0^{[0,2]}(3,2,2)m_cm_b^2$$

$$C_1^{V-A} = -40\hat{I}_4^{[0,1]}(2,3,1)m_b + 20\hat{I}_4^{[0,2]}(3,2,2)m_b - 40\hat{I}_3(2,2,1)m_b - 20\hat{I}_1(1,2,2)m_b$$
$$- 20\hat{I}_1(2,1,2)m_b - 40\hat{I}_3^{[0,1]}(2,3,1)m_b - 20\hat{I}_3^{[0,1]}(3,2,1)m_b - 20\hat{I}_3(2,1,2)m_b$$
$$- 20\hat{I}_3^{[0,1]}(2,2,2)m_b - 20\hat{I}_3(1,2,2)m_b - 20\hat{I}_4(1,2,2)m_b - 10\hat{I}_1^{[0,1]}(2,3,1)m_b$$
$$+ 5\hat{I}_1^{[0,2]}(3,2,2)m_b - 5\hat{I}_2(3,2,2)m_3^5 - 5\hat{I}_0(3,2,2)m_c^5 - 20\hat{I}_3(3,2,2)m_c^5$$
$$- 15\hat{I}_1(3,2,2)m_c^5 - 45\hat{I}_1(3,2,1)m_c^3 - 20\hat{I}_4(3,1,2)m_c^3 - 20\hat{I}_2(3,2,1)m_c^3$$
$$- 15\hat{I}_2(4,1,1)m_c^3 - 25\hat{I}_0(3,2,1)m_c^3 - 40\hat{I}_4(3,2,1)m_c^3 - 10\hat{I}_2(2,2,2)m_c^3$$
$$- 45\hat{I}_1(4,1,1)m_c^3 - 20\hat{I}_4(3,2,2)m_c^5 - 5\hat{I}_0(3,1,2)m_c^3 - 40\hat{I}_3(2,2,2)m_c^3$$
$$+ 10\hat{I}_0^{[0,1]}(3,2,2)m_c^3 - 30\hat{I}_1(2,2,2)m_c^3 + 40\hat{I}_3^{[0,1]}(3,2,2)m_c^3 - 60\hat{I}_3(4,1,1)m_c^3$$
$$- 15\hat{I}_0(4,1,1)m_c^3 - 5\hat{I}_2(3,1,2)m_c^3 - 10\hat{I}_0(2,2,2)m_c^3 - 20\hat{I}_1(3,1,2)m_c^3$$
$$- 40\hat{I}_3(3,2,1)m_c^3 + 30\hat{I}_1^{[0,1]}(3,2,2)m_c^3 + 40\hat{I}_4^{[0,1]}(3,2,2)m_c^3 - 60\hat{I}_4(4,1,1)m_c^3$$
$$- 20\hat{I}_3(3,2,2)m_c^2m_b^3 + 20\hat{I}_4(2,2,2)m_c^2m_b - 20\hat{I}_0(2,3,1)m_c^2m_b$$
$$+ 40\hat{I}_4(3,2,1)m_c^2m_b - 40\hat{I}_4^{[0,1]}(3,2,2)m_c^2m_b - 20\hat{I}_2(2,3,1)m_c^2m_b$$
$$+ 60\hat{I}_4(4,1,1)m_c^2m_b + 60\hat{I}_3(4,1,1)m_c^2m_b + 20\hat{I}_3(3,1,2)m_c^2m_b$$
$$+ 20\hat{I}_3(2,2,2)m_c^2m_b + 5\hat{I}_1(3,2,2)m_c^4m_b + 20\hat{I}_3(3,2,2)m_c^4m_b$$
$$+ 20\hat{I}_4(3,2,2)m_c^4m_b + 5\hat{I}_0(3,2,2)m_c^3m_b^2 + 20\hat{I}_4(3,2,2)m_c^3m_b^2 + 5\hat{I}_2(3,2,2)m_c^3m_b^2$$
$$+ 15\hat{I}_1(3,2,2)m_c^3m_b^2 + 20\hat{I}_3(3,2,2)m_c^3m_b^2 - 20\hat{I}_4(3,2,2)m_c^2m_b^3 - 5\hat{I}_1(3,2,2)m_c^2m_b^3$$
$$+ 15\hat{I}_2(3,2,1)m_c^2m_b + 15\hat{I}_1(4,1,1)m_c^2m_b + 5\hat{I}_0(3,2,1)m_c^2m_b + 10\hat{I}_1(3,1,2)m_c^2m_b$$
$$- 50\hat{I}_1(2,3,1)m_c^2m_b - 10\hat{I}_1^{[0,1]}(3,2,2)m_c^2m_b + 35\hat{I}_1(3,2,1)m_c^2m_b$$
$$+ 20\hat{I}_4(3,1,2)m_3^2m_b - 40\hat{I}_3^{[0,1]}(3,2,2)m_c^2m_b - 40\hat{I}_3(2,3,1)m_c^2m_b$$
$$- 20\hat{I}_3(3,1,2)m_c^3 + 10\hat{I}_2^{[0,1]}(3,2,2)m_c^3 + 40\hat{I}_3(2,3,1)m_b^3 + 20\hat{I}_3(2,2,2)m_b^3$$
$$- 5\hat{I}_1^{[0,1]}(3,2,2)m_b^3 + 40\hat{I}_4(2,3,1)m_b^3 - 20\hat{I}_4^{[0,1]}(3,2,2)m_b^3 - 120\hat{I}_3(1,4,1)m_b^3$$
$$+ 20\hat{I}_3(3,2,1)m_b^3 - 120\hat{I}_4(1,4,1)m_b^3 + 20\hat{I}_4(3,2,1)m_b^3 + 10\hat{I}_1(2,3,1)m_b^3$$
$$- 30\hat{I}_1(1,4,1)m_b^3 - 40\hat{I}_4(2,2,2)m_c^3 + 40\hat{I}_3(3,2,1)m_c^2m_b - 40\hat{I}_4(2,3,1)m_c^2m_b$$
$$+ 20\hat{I}_4^{[0,1]}(3,2,2)m_cm_b^2 + 5\hat{I}_0^{[0,1]}(3,2,2)m_cm_b^2 - 20\hat{I}_4(2,2,2)m_cm_b^2$$
$$- 30\hat{I}_1(3,2,1)m_cm_b^2 + 90\hat{I}_1(1,4,1)m_cm_b^2 + 120\hat{I}_3(1,4,1)m_3m_b^2$$
$$+ 20\hat{I}_3^{[0,1]}(3,2,2)m_cm_b^2 + 120\hat{I}_4(1,4,1)m_cm_b^2 + 10\hat{I}_1(3,2,1)m_b^3$$
$$- 20\hat{I}_3^{[0,1]}(3,2,2)m_b^3 + 40\hat{I}_3(3,1,1)m_c - 5\hat{I}_0(2,2,1)m_c + 10\hat{I}_0^{[0,1]}(2,2,2)m_c$$
$$+ 20\hat{I}_4(2,1,2)m_c + 40\hat{I}_3^{[0,1]}(3,2,1)m_c + 20\hat{I}_4^{[0,1]}(3,1,2)m_c + 10\hat{I}_2^{[0,1]}(2,2,2)m_c$$
$$+ 20\hat{I}_3(2,2,1)m_c - 20\hat{I}_4^{[0,2]}(3,2,2)m_c + 40\hat{I}_4^{[0,1]}(2,2,2)m_c + 5\hat{I}_2(3,1,1)m_c$$
$$+ 20\hat{I}_4(2,2,2)m_b^3 + 30\hat{I}_0(1,4,1)m_cm_b^2 + 5\hat{I}_1(3,1,2)m_cm_b^2 - 15\hat{I}_0(3,2,1)m_cm_b^2$$
$$+ 5\hat{I}_2^{[0,1]}(3,2,2)m_cm_b^2 - 10\hat{I}_2(3,2,1)m_cm_b^2 + 30\hat{I}_2(1,4,1)m_cm_b^2$$

$$
\begin{aligned}
&- 20\hat{I}_4(3,2,1)m_c m_b^2 + 15\hat{I}_1^{[0,1]}(3,2,2)m_c m_b^2 - 10\hat{I}_1(2,2,2)m_c m_b^2\\
&- 20\hat{I}_3(3,2,1)m_c m_b^2 - 20\hat{I}_3(2,2,2)m_c m_b^2 - 15\hat{I}_1^{[0,2]}(3,2,2)m_c - 10\hat{I}_1(2,1,2)m_c\\
&- 5\hat{I}_2^{[0,2]}(3,2,2)m_c + 5\hat{I}_1(2,2,1)m_c + 40\hat{I}_4^{[0,1]}(3,2,1)m_c + 10\hat{I}_2^{[0,1]}(3,2,1)m_c\\
&+ 15\hat{I}_0^{[0,1]}(3,1,2)m_c + 20\hat{I}_3^{[0,1]}(3,1,2)m_c + 35\hat{I}_1^{[0,1]}(3,2,1)m_c + 30\hat{I}_1^{[0,1]}(3,1,2)m_c\\
&- 5\hat{I}_0^{[0,2]}(3,2,2)m_c + 40\hat{I}_3^{[0,1]}(2,2,2)m_c + 20\hat{I}_3(2,1,2)m_c - 15\hat{I}_0(2,1,2)m_c\\
&- 40\hat{I}_3^{[0,1]}(3,1,2)m_b - 40\hat{I}_4(3,1,1)m_b - 10\hat{I}_1^{[0,1]}(3,2,1)m_b - 40\hat{I}_4(2,2,1)m_b\\
&+ 20\hat{I}_3^{[0,2]}(3,2,2)m_b - 40\hat{I}_3(1,3,1)m_b - 40\hat{I}_4^{[0,1]}(3,1,2)m_b + 10\hat{I}_1(1,3,1)m_b\\
&- 10\hat{I}_0(2,2,1)m_b - 20\hat{I}_4^{[0,1]}(2,2,2)m_b - 10\hat{I}_2(2,2,1)m_b - 10\hat{I}_1^{[0,1]}(3,1,2)m_b\\
&+ 10\hat{I}_0(1,3,1)m_b - 20\hat{I}_4^{[0,1]}(3,2,1)m_b - 20\hat{I}_3^{[0,2]}(3,2,2)m_c - 10\hat{I}_0(3,1,1)m_c\\
&+ 40\hat{I}_4(3,1,1)m_c - 15\hat{I}_2(2,1,2)m_c + 20\hat{I}_4(2,2,1)m_c + 15\hat{I}_1(3,1,1)m_c\\
&+ 15\hat{I}_2^{[0,1]}(3,1,2)m_c + 15\hat{I}_0^{[0,1]}(3,2,1)m_c - 40\hat{I}_3(3,1,1)m_b - 20\hat{I}_4(2,1,2)m_b\\
&+ 10\hat{I}_2(1,3,1)m_b - 30\hat{I}_1(2,2,1)m_b - 40\hat{I}_4(1,3,1)m_b + 30\hat{I}_1^{[0,1]}(2,2,2)m_c
\end{aligned}
$$

$$
\begin{aligned}
C_2^{V-A} = {}& 15\hat{I}_2(4,1,1)m_c^2 m_b - 40\hat{I}_3^{[0,1]}(3,2,2)m_c^2 m_b - 40\hat{I}_4(3,2,1)m_c^2 m_b\\
&- 10\hat{I}_2(2,3,1)m_3^2 m_b + 20\hat{I}_3(3,1,2)m_c^2 m_b + 60\hat{I}_3(4,1,1)m_c^2 m_b\\
&- 20\hat{I}_4(2,2,2)m_c^2 m_b + 40\hat{I}_3(3,2,1)m_c^2 m_b + 40\hat{I}_4^{[0,1]}(3,2,2)m_c^2 m_b\\
&- 60\hat{I}_4(4,1,1)m_c^2 m_b + 40\hat{I}_4(2,3,1)m_c^2 m_b + 20\hat{I}_3(2,2,2)m_c^2 m_b\\
&- 5\hat{I}_0(3,2,1)m_c^2 m_b + 15\hat{I}_1(3,2,1)m_c^2 m_b - 20\hat{I}_1(2,3,1)m_c^2 m_b\\
&- 40\hat{I}_3(2,3,1)m_c^2 m_b - 20\hat{I}_4(3,1,2)m_c^2 m_b + 10\hat{I}_2^{[0,1]}(3,2,2)m_c^3\\
&+ 60\hat{I}_4(4,1,1)m_c^3 - 20\hat{I}_3(3,1,2)m_c^3 + 40\hat{I}_4(2,2,2)m_c^3 - 15\hat{I}_1(4,1,1)m_c^3\\
&- 40\hat{I}_3(3,2,1)m_c^3 - 40\hat{I}_3(2,2,2)m_c^3 - 15\hat{I}_2(4,1,1)m_c^3 - 5\hat{I}_2(3,2,1)m_c^3\\
&+ 10\hat{I}_1^{[0,1]}(3,2,2)m_c^3 + 5\hat{I}_0(3,1,2)m_c^3 - 10\hat{I}_0^{[0,1]}(3,2,2)m_c^3 - 10\hat{I}_1(2,2,2)m_c^3\\
&+ 10\hat{I}_0(2,2,2)m_c^3 - 10\hat{I}_2(3,1,2)m_c^3 - 5\hat{I}_1(3,1,2)m_c^3 + 15\hat{I}_0(4,1,1)m_c^3\\
&- 20\hat{I}_1(3,2,1)m_c^3 + 20\hat{I}_4^{[0,1]}(3,2,2)m_b^3 - 30\hat{I}_2(1,4,1)m_b^3 - 20\hat{I}_3^{[0,1]}(3,2,2)m_b^3\\
&+ 40\hat{I}_3(2,3,1)m_b^3 + 5\hat{I}_2(3,2,1)m_c^2 m_b - 20\hat{I}_4^{[0,1]}(3,2,2)m_c m_b^2\\
&+ 5\hat{I}_2(3,1,2)m_c m_b^2 - 20\hat{I}_3(3,2,1)m_c m_b^2 + 20\hat{I}_4(2,2,2)m_c {m_b}^2\\
&- 5\hat{I}_0^{[0,1]}(3,2,2)m_c m_b^2 - 10\hat{I}_2(3,2,1)m_c m_b^2 + 15\hat{I}_0(3,2,1)m_c m_b^2\\
&+ 20\hat{I}_3^{[0,1]}(3,2,2)m_c m_b^2 - 10\hat{I}_2(2,2,2)m_c m_b^2 - 30\hat{I}_0(1,4,1)m_c m_b^2\\
&+ 120\hat{I}_3(1,4,1)m_c m_b^2 + 5\hat{I}_1^{[0,1]}(3,2,2)m_c m_b^2 + 30\hat{I}_1(1,4,1)m_c m_b^2\\
&- 120\hat{I}_4(1,4,1)m_c m_b^2 + 5\hat{I}_2^{[0,1]}(3,2,2)m_c m_b^2 - 20\hat{I}_3(2,2,2)m_c m_b^2\\
&+ 20\hat{I}_4(3,2,1)m_c m_b^2 - 10\hat{I}_1(3,2,1)m_c m_b^2 + 30\hat{I}_2(1,4,1)m_c m_b^2 + 20\hat{I}_3(2,2,2)m_b^3\\
&- 20\hat{I}_4(2,2,2)m_b^3 - 5\hat{I}_2^{[0,1]}(3,2,2)m_b^3 - 40\hat{I}_4(2,3,1)m_b^3 + 120\hat{I}_4(1,4,1)m_b^3\\
&- 20\hat{I}_4(3,2,1)m_b^3 + 10\hat{I}_2(2,3,1)m_b^3 + 10\hat{I}_2(3,2,1)m_b^3 - 120\hat{I}_3(1,4,1)m_b^3\\
&+ 20\hat{I}_3(3,2,1)m_b^3 + 20\hat{I}_4^{[0,2]}(3,2,2)m_c + 10\hat{I}_1^{[0,1]}(3,2,1)m_c - 40\hat{I}_4^{[0,1]}(2,2,2)m_c\\
&+ 5\hat{I}_0(2,2,1)m_c + 15\hat{I}_1^{[0,1]}(3,1,2)m_c - 5C1_2(3,2,2)m_c - 15\hat{I}_0^{[0,1]}(3,2,1)m_c\\
&+ 5\hat{I}_0^{[0,2]}(3,2,2)m_c - 40\hat{I}_4(3,1,1)m_c + 15\hat{I}_0(2,1,2)m_c + 5\hat{I}_2(3,1,1)m_c\\
&+ 40\hat{I}_3(3,1,1)m_c + 5\hat{I}_1(3,1,1)m_c - 20\hat{I}_4^{[0,1]}(3,1,2)m_c + 10\hat{I}_0(3,1,1)m_c\\
&+ 20\hat{I}_3(2,2,1)m_c + 15\hat{I}_2^{[0,1]}(3,2,1)m_c - 20\hat{I}_4(2,2,1)m_c + 20\hat{I}_2(2,1,2)m_c
\end{aligned}
$$

$$
\begin{aligned}
&+10\hat{I}_1^{[0,1]}(2,2,2)m_c+20\hat{I}_3^{[0,1]}(3,1,2)m_c-20\hat{I}_4(2,1,2)m_c+20\hat{I}_3(2,1,2)m_c\\
&-40\hat{I}_4^{[0,1]}(3,2,1)m_c+40\hat{I}_3^{[0,1]}(3,2,1)m_c-5\hat{I}_1^{[0,2]}(3,2,2)m_c+40\hat{I}_3^{[0,1]}(2,2,2)m_c\\
&+5\hat{I}_2(2,2,1)m_c+10\hat{I}_2^{[0,1]}(2,2,2)m_c-15\hat{I}_1(2,1,2)m_c-15\hat{I}_0^{[0,1]}(3,1,2)m_c\\
&-10\hat{I}_0^{[0,1]}(2,2,2)m_c-20\hat{I}_3^{[0,2]}(3,2,2)m_c+10\hat{I}_1(1,3,1)m_b+20\hat{I}_4(1,2,2)m_b\\
&-40\hat{I}_3^{[0,1]}(3,1,2)m_b+40\hat{I}_4^{[0,1]}(3,1,2)m_b-20\hat{I}_2(2,1,2)m_b+20\hat{I}_4^{[0,1]}(3,2,1)m_b\\
&-10\hat{I}_2^{[0,1]}(3,2,1)m_b-20\hat{I}_3(1,2,2)m_b-10\hat{I}_2^{[0,1]}(2,3,1)m_b-40\hat{I}_3(1,3,1)m_b\\
&+5\hat{I}_2^{[0,2]}(3,2,2)m_b+10\hat{I}_0(2,2,1)m_b-20\hat{I}_3(2,1,2)m_b-10\hat{I}_1(2,2,1)m_b\\
&-10\hat{I}_2(2,2,1)m_b-10\hat{I}_0(1,3,1)m_b-20\hat{I}_4^{[0,2]}(3,2,2)m_b-40\hat{I}_3^{[0,1]}(2,3,1)m_b\\
&-20\hat{I}_2(1,2,2)m_b+20\hat{I}_4(2,1,2)m_b-40\hat{I}_3(3,1,1)m_b-10\hat{I}_2(1,3,1)m_b\\
&-10\hat{I}_2^{[0,1]}(3,1,2)m_b-20\hat{I}_3^{[0,1]}(2,2,2)m_b+40\hat{I}_4(1,3,1)m_b+40\hat{I}_4(3,1,1)m_b\\
&+20\hat{I}_3^{[0,2]}(3,2,2)m_b-40\hat{I}_3(2,2,1)m_b+20\hat{I}_4^{[0,1]}(2,2,2)m_b+40\hat{I}_4^{[0,1]}(2,3,1)m_b\\
&-20\hat{I}_3^{[0,1]}(3,2,1)m_b+40\hat{I}_4(2,2,1)m_b+25\hat{I}_0(3,2,1)m_c^3-60\hat{I}_3(4,1,1)m_c^3\\
&+10\hat{I}_2(3,1,2)m_c^2m_b+20\hat{I}_4(3,2,2)m_c^5+5\hat{I}_0(3,2,2)m_c^5-5\hat{I}_2(3,2,2)m_c^5\\
&-20\hat{I}_3(3,2,2)m_c^5-5\hat{I}_1(3,2,2){m_c}^5-40\hat{I}_4^{[0,1]}(3,2,2)m_c^3-10\hat{I}_2(2,2,2)m_c^3\\
&+40\hat{I}_4(3,2,1)m_c^3+40\hat{I}_3^{[0,1]}(3,2,2)m_c^3+20\hat{I}_4(3,1,2)m_c^3+5\hat{I}_2(3,2,2)m_c^4m_b\\
&-20\hat{I}_4(3,2,2)m_c^4m_b+20\hat{I}_3(3,2,2)m_c^4m_b+5\hat{I}_1(3,2,2)m_c^3m_b^2-5\hat{I}_0(3,2,2)m_c^3m_b^2\\
&+20\hat{I}_3(3,2,2)m_c^3m_b^2-20\hat{I}_4(3,2,2)m_c^3m_b^2+5\hat{I}_2(3,2,2)m_c^3m_b^2\\
&-5\hat{I}_2(3,2,2)m_c^2m_b^3+20\hat{I}_4(3,2,2)m_c^2m_b^3-20\hat{I}_3(3,2,2)m_c^2m_b^3\\
&-10\hat{I}_2^{[0,1]}(3,2,2){m_c}^2m_b+20\hat{I}_0(2,3,1)m_c^2m_b
\end{aligned}
$$

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\begin{aligned}
C_V^{T-PT}={}&10\hat{I}_1(3,2,2)m_b^4m_c^2-20\hat{I}_1(1,1,2)+10\hat{I}_0^{[0,1]}(2,2,1)-10\hat{I}_2^{[0,2]}(3,2,1)\\
&-10\hat{I}_0(3,2,2)m_b^3m_c^3-10\hat{I}_1(3,2,2)m_b^3m_c^3-10\hat{I}_1(1,2,1)-10\hat{I}_0(1,1,2)\\
&-60\hat{I}_1(1,4,1)m_b^3m_c-10\hat{I}_0^{[0,1]}(3,2,2)m_b^3m_c-60\hat{I}_0(1,4,1)m_b^3m_c\\
&+20\hat{I}_2(3,2,1)m_b^3m_c+20\hat{I}_1(3,2,1)m_b^3m_c-10\hat{I}_2^{[0,1]}(3,2,2)m_b^3m_c\\
&-10\hat{I}_1^{[0,1]}(3,2,2)m_b^3m_c-60\hat{I}_2(1,4,1)m_b^3m_c+20\hat{I}_2(2,3,1)m_b^3m_c\\
&-10\hat{I}_2(3,2,2)m_b^3{m_c}^3+10\hat{I}_2(3,2,2)m_bm_c^5+10\hat{I}_1(3,2,2)m_bm_c^5\\
&+10\hat{I}_0(3,2,2)m_bm_c^5+20\hat{I}_0(3,2,1)m_b^3m_c-20\hat{I}_1(2,3,1)m_b^3m_c\\
&+10\hat{I}_1(3,1,2)m_b^3m_c+20\hat{I}_0(2,3,1)m_b^3m_c-10\hat{I}_1(3,2,2)m_b^2{m_c}^4\\
&+10\hat{I}_1^{[0,1]}(3,2,2)m_b^4-20\hat{I}_1(3,2,1)m_b^4+60\hat{I}_1(1,4,1)m_b^4-10\hat{I}_0(3,1,2)m_c^4\\
&-10\hat{I}_1(3,1,2)m_c^4+10\hat{I}_1(3,2,1)m_c^4+10\hat{I}_0(3,2,1)m_c^4+10\hat{I}_2^{[0,1]}(3,2,1)m_b^2\\
&+60\hat{I}_1(1,3,1)m_b^2+20\hat{I}_1^{[0,1]}(3,2,1)m_b^2-10\hat{I}_1(2,2,2)m_b^4+20\hat{I}_0(2,3,1)m_bm_c^3\\
&-20\hat{I}_2^{[0,1]}(3,2,2)m_bm_c^3+20\hat{I}_1(2,2,2)m_bm_c^3+10\hat{I}_2(3,2,1)m_bm_c^3\\
&+20\hat{I}_2(2,3,1)m_bm_c^3+10\hat{I}_1(3,2,1)m_bm_c^3+20\hat{I}_2(2,2,2)m_bm_c^3\\
&+20\hat{I}_1(2,3,1)m_bm_c^3+30\hat{I}_0(4,1,1)m_bm_c^3+20\hat{I}_1^{[0,1]}(3,2,2)m_b^2m_c^2\\
&+10\hat{I}_2(3,2,1)m_b^2m_c^2-20\hat{I}_1(3,2,1)m_b^2m_c^2-30\hat{I}_1(4,1,1)m_b^2m_c^2\\
&+20\hat{I}_0(3,2,1)m_bm_c^3+30\hat{I}_1(4,1,1)m_bm_c^3+30\hat{I}_2(4,1,1)m_bm_c^3\\
&+20\hat{I}_0(2,2,2)m_bm_c^3-20\hat{I}_1(2,2,2)m_b^2m_c^2-40\hat{I}_2(1,3,1)m_bm_c\\
&-30\hat{I}_2(2,2,1)m_bm_c-70\hat{I}_1(2,2,1)m_bm_c-10\hat{I}_2^{[0,1]}(3,2,1)m_bm_c
\end{aligned}
$$

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\begin{aligned}
&- 20\hat{I}_0^{[0,1]}(2,2,2)m_bm_c - 20\hat{I}_2^{[0,1]}(2,2,2)m_bm_c + 20\hat{I}_0(1,2,2)m_bm_c \\
&+ 10\hat{I}_0^{[0,2]}(3,2,2)m_bm_c + 10\hat{I}_1^{[0,2]}(3,2,2)m_bm_c - 20\hat{I}_0^{[0,1]}(3,2,1)m_bm_c \\
&+ 20\hat{I}_2(1,2,2)m_bm_c - 20\hat{I}_1^{[0,1]}(2,3,1)m_bm_c - 20\hat{I}_0(2,2,1)m_bm_c \\
&- 20\hat{I}_2^{[0,1]}(3,1,2)m_bm_c + 20\hat{I}_2(2,1,2)m_bm_c - 40\hat{I}_0(1,3,1)m_bm_c \\
&- 20\hat{I}_0^{[0,1]}(3,1,2)m_bm_c - 20\hat{I}_2^{[0,1]}(2,3,1)m_bm_c - 10\hat{I}_1^{[0,1]}(3,2,1)m_bm_c \\
&+ 20\hat{I}_0(2,1,2)m_bm_c + 10\hat{I}_2^{[0,2]}(3,2,2)m_bm_c - 10\hat{I}_0(1,2,1) - 20\hat{I}_1(1,2,2)m_b^2 \\
&+ 10\hat{I}_0(2,2,1)m_b^2 + 20\hat{I}_1^{[0,1]}(3,1,2)m_b^2 - 30\hat{I}_1(2,1,2)m_b^2 + 20\hat{I}_1^{[0,1]}(2,2,2)m_b^2 \\
&+ 30\hat{I}_2(2,2,1)m_b^2 - 10\hat{I}_1^{[0,2]}(3,2,2)m_b^2 + 10\hat{I}_0^{[0,1]}(3,1,2)m_c^2 - 20\hat{I}_0(2,1,2)m_c^2 \\
&- 10\hat{I}_0^{[0,1]}(3,2,1)m_c^2 - 20\hat{I}_1(2,1,2)m_c^2 + 60\hat{I}_1(2,2,1)m_b^2 - 30\hat{I}_2(1,2,1) \\
&- 20\hat{I}_0^{[0,1]}(3,2,2)m_bm_c^3 + 20\hat{I}_1(2,1,2)m_bm_c - 20\hat{I}_0^{[0,1]}(2,3,1)m_bm_c \\
&+ 20\hat{I}_1(1,2,2)m_bm_c - 10\hat{I}_1(3,1,1)m_bm_c - 20\hat{I}_1^{[0,1]}(3,1,2)m_bm_c \\
&- 20\hat{I}_1^{[0,1]}(2,2,2)m_bm_c - 40\hat{I}_1(1,3,1)m_bm_c - 20\hat{I}_1^{[0,1]}(3,2,2)m_bm_c^3 \\
&- 10\hat{I}_1^{[0,1]}(3,2,1)m_c^2 + 10\hat{I}_1^{[0,1]}(3,1,2)m_c^2 - 20\hat{I}_2(2,2,1)m_c^2 + 10\hat{I}_0(3,1,1)m_c^2 \\
&+ 10\hat{I}_2^{[0,1]}(3,2,1)m_c^2 - 10\hat{I}_2^{[0,1]}(3,1,2)m_c^2 + 10\hat{I}_2(3,1,1)m_c^2
\end{aligned}
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\begin{aligned}
C_0^{T-PT} = &-15\hat{I}_0(2,2,2)m_c^4m_b^2 - 15\hat{I}_0(4,1,1)m_c^4m_b^2 + 5\hat{I}_0(3,1,2)m_c^5m_b - 5\hat{I}_0(3,2,1)m_c^5m_b \\
&- 5\hat{I}_0(3,2,2)m_c^3m_b^5 + 5\hat{I}_0(3,2,2)m_c^4m_b^4 + 5\hat{I}_0(3,2,2)m_c^5m_b^3 - 5\hat{I}_0(3,2,2)m_c^6m_b^2 \\
&+ 30\hat{I}_0^{[0,1]}(2,2,2)m_c^2m_b^2 - 15\hat{I}_0(2,1,2){m_c}^2m_b^2 - 15\hat{I}_0^{[0,2]}(3,2,2)m_c^2m_b^2 \\
&- 20\hat{I}_0(2,2,1)m_c^3m_b + 10\hat{I}_0^{[0,1]}(3,2,1)m_c^3m_b + 10\hat{I}_0(2,1,2)m_c^3m_b \\
&- 30\hat{I}_0(1,4,1)m_cm_b^5 - 5\hat{I}_0^{[0,1]}(3,2,2)m_cm_b^5 + 10\hat{I}_0(3,2,1)m_cm_b^5 \\
&+ 10\hat{I}_0(2,3,1)m_cm_b^5 - 10\hat{I}_0(3,2,1)m_c^2m_b^4 + 30\hat{I}_0(1,4,1)m_c^2m_b^4 \\
&+ 15\hat{I}_0(4,1,1)m_c^3m_b^3 - 10\hat{I}_0^{[0,1]}(3,2,2)m_c^3m_b^3 - 10\hat{I}_0(2,3,1)m_c^3m_b^3 \\
&+ 10\hat{I}_0(2,2,2)m_c^3m_b^3 + 15\hat{I}_0^{[0,1]}(3,2,2)m_c^4m_b^2 + 10\hat{I}_0(1,2,2)m_cm_b^3 \\
&+ 15\hat{I}_0(2,1,2)m_cm_b^3 - 20\hat{I}_0^{[0,1]}(3,1,2)m_cm_b^3 + 10\hat{I}_0^{[0,1]}(2,3,1)m_cm_b^3 \\
&- 5\hat{I}_0(2,2,1)m_cm_b^3 - 10\hat{I}_0^{[0,1]}(2,2,2)m_cm_b^3 - 20\hat{I}_0(1,2,2)m_c^2m_b^2 \\
&+ 15\hat{I}_0^{[0,1]}(3,2,1)m_c^2m_b^2 - 5\hat{I}_0(3,1,1)m_c^2m_b^2 + 30\hat{I}_0(1,3,1)m_c^2m_b^2 \\
&+ 15\hat{I}_0^{[0,1]}(3,1,2)m_c^2m_b^2 - 30\hat{I}_0(2,1,1)m_cm_b + 10\hat{I}_0^{[0,1]}(3,1,1)m_cm_b \\
&- 5\hat{I}_0^{[0,2]}(3,2,1)m_cm_b - 10\hat{I}_0^{[0,1]}(2,1,2)m_cm_b - 50\hat{I}_0(1,2,1)m_cm_b \\
&+ 10\hat{I}_0(1,1,2)m_cm_b + 5\hat{I}_0^{[0,2]}(3,1,2)m_cm_b + 20\hat{I}_0^{[0,1]}(2,2,1)m_cm_b \\
&- 70\hat{I}_0(1,3,1)m_cm_b^3 + 5\hat{I}_0^{[0,2]}(3,2,2)m_cm_b^3 + 5\hat{I}_0(3,1,1)m_c{m_b}^3 \\
&- 20\hat{I}_0^{[0,1]}(1,2,1) - 5\hat{I}_0^{[0,2]}(3,2,1)m_b^2 - 15\hat{I}_0(1,1,2)m_b^2 + 30\hat{I}_0^{[0,1]}(2,1,2)m_b^2 \\
&- 15\hat{I}_0^{[0,2]}(3,1,2)m_b^2 - 30\hat{I}_0^{[0,1]}(1,3,1)m_b^2 + 10\hat{I}_0(1,2,1)m_b^2 - 15\hat{I}_0^{[0,2]}(2,1,2) \\
&+ 20\hat{I}_0^{[0,1]}(1,2,2)m_b^2 - 15\hat{I}_0^{[0,2]}(2,2,2)m_b^2 + 20\hat{I}_0(2,1,1)m_b^2 + 30\hat{I}_0^{[0,1]}(2,1,2)m_c^2 \\
&- 30\hat{I}_0^{[0,1]}(2,2,1)m_c^2 + 10\hat{I}_0^{[0,1]}(3,1,1)m_b^2 - 5\hat{I}_0^{[0,1]}(2,2,1)m_b^2 + 20\hat{I}_0(1,2,1)m_c^2 \\
&- 15\hat{I}_0^{[0,2]}(3,1,2)m_c^2 + 15\hat{I}_0^{[0,2]}(3,2,1)m_c^2 - 20\hat{I}_0(1,1,2)m_c^2 + 15\hat{I}_0(2,1,1)m_c^2 \\
&- 30\hat{I}_0^{[0,1]}(1,4,1)m_b^4 + 15\hat{I}_0(1,1,1) + 10\hat{I}_0(2,2,1)m_b^4 + 10\hat{I}_0^{[0,1]}(2,2,2)m_b^4 \\
&+ 10\hat{I}_0^{[0,1]}(3,2,1)m_b^4 - 5\hat{I}_0^{[0,2]}(3,2,2)m_b^4 + 15\hat{I}_0^{[0,1]}(3,1,2)m_c^4 - 5\hat{I}_0(1,2,2)m_b^4
\end{aligned}
$$

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\begin{aligned}
&- 15\hat{I}_0(2,1,2)m_c^4 - 15\hat{I}_0^{[0,1]}(3,2,1)m_c^4 - 5\hat{I}_0(3,1,2)m_c^6 + 5\hat{I}_0(3,2,1)m_c^6\\
&+ 15\hat{I}_0(2,2,1)m_c^4 + 20\hat{I}_0^{[0,1]}(1,1,2) + 15\hat{I}_0^{[0,2]}(2,2,1)\\
C_1^{T-PT} =& +10\hat{I}_0(3,1,1)m_c m_b + 10\hat{I}_1^{[0,1]}(2,2,2)m_c m_b - 10\hat{I}_2^{[0,1]}(2,2,2)m_c m_b\\
&- 35\hat{I}_2(2,2,1)m_c m_b - 10\hat{I}_0^{[0,1]}(2,2,2)m_c m_b - 10\hat{I}_0(1,1,2) + 20\hat{I}_1(1,2,2)m_c^2\\
&+ 15\hat{I}_0^{[0,1]}(4,1,1)m_c^2 + 25\hat{I}_2^{[0,1]}(3,1,2)m_c^2 - 15\hat{I}_0(2,1,2)m_c^2 - 5\hat{I}_1(3,1,1)m_c^2\\
&- 15\hat{I}_1^{[0,1]}(3,1,2)m_c^2 + 30\hat{I}_2^{[0,1]}(2,2,2)m_c^2 + 15\hat{I}_1(2,1,2)m_c^2 + 30\hat{I}_0^{[0,1]}(2,2,2)m_c^2\\
&+ 5\hat{I}_2^{[0,1]}(3,2,1)m_c^2 - 15\hat{I}_1^{[0,1]}(3,2,1)m_c^2 + 15\hat{I}_1^{[0,2]}(3,2,2)m_c^2 - 15\hat{I}_0^{[0,2]}(3,2,2)m_c^2\\
&+ 10\hat{I}_2(2,2,1)m_c^2 + 10\hat{I}_2(3,1,1)m_c^2 - 5\hat{I}_0(3,2,2)m_c^3 m_b^3 - 5\hat{I}_2(3,2,2)m_c^3 m_b^3\\
&+ 5\hat{I}_2(3,2,2)m_c^2 m_b^4 - 15\hat{I}_1(4,1,1)m_c^3 m_b + 10\hat{I}_2(2,2,2)m_c^3 m_b - 15\hat{I}_0^{[0,2]}(2,2,2)\\
&- 15\hat{I}_1(2,1,2)m_c m_b - 5\hat{I}_2^{[0,1]}(3,2,1)m_c m_b + 5\hat{I}_0^{[0,2]}(3,2,2)m_c m_b\\
&- 10\hat{I}_0^{[0,1]}(3,2,1)m_c m_b - 15\hat{I}_0^{[0,1]}(3,1,2)m_c m_b + 15\hat{I}_1^{[0,1]}(3,1,2)m_c m_b\\
&+ 10\hat{I}_2(1,2,2)m_c m_b - 10\hat{I}_1^{[0,1]}(2,3,1)m_c m_b + 5\hat{I}_1(3,1,1)m_c m_b\\
&+ 30\hat{I}_2^{[0,1]}(2,1,2) - 20\hat{I}_1^{[0,1]}(2,1,2) + 10\hat{I}_1^{[0,2]}(3,1,2) - 5\hat{I}_2^{[0,1]}(3,1,1)\\
&- 10\hat{I}_1(1,2,2)m_c m_b + 10\hat{I}_0(1,2,2)m_c m_b - 15\hat{I}_2^{[0,1]}(3,1,2)m_c m_b\\
&- 15\hat{I}_2^{[0,2]}(3,1,2) + 30\hat{I}_2(1,3,1)m_b^2 + 20\hat{I}_2(2,2,1)m_b^2 + 5\hat{I}_2(3,1,1)m_b^2\\
&- 5\hat{I}_0^{[0,2]}(3,2,2)m_b^2 - 10\hat{I}_2^{[0,2]}(3,2,2)m_b^2 + 10\hat{I}_0^{[0,1]}(2,2,2)m_b^2 + 5\hat{I}_0(2,2,1)m_b^2\\
&- 30\hat{I}_0^{[0,1]}(1,4,1)m_b^2 - 10\hat{I}_1(2,2,1)m_b^2 - 5\hat{I}_0(1,2,2)m_b^2 + 5\hat{I}_1^{[0,2]}(3,2,2)m_b^2\\
&+ 10\hat{I}_2^{[0,1]}(3,2,1)m_b^2 - 10\hat{I}_1^{[0,1]}(2,2,2)m_b^2 - 15\hat{I}_2(1,2,2)m_b^2 + 30\hat{I}_1^{[0,1]}(1,4,1)m_b^2\\
&- 5\hat{I}_1^{[0,1]}(2,2,1) - 5\hat{I}_2^{[0,1]}(2,2,1) + 20\hat{I}_0^{[0,1]}(1,2,2) - 15\hat{I}_2(2,1,2)m_b^2\\
&- 30\hat{I}_2^{[0,1]}(1,4,1)m_b^2 + 15\hat{I}_2^{[0,1]}(3,1,2)m_b^2 + 20\hat{I}_2^{[0,1]}(2,2,2)m_b^2\\
&- 10\hat{I}_1^{[0,1]}(3,2,1)m_b^2 + 15\hat{I}_0^{[0,1]}(3,2,1)m_b^2 + 5\hat{I}_1(1,2,2)m_b^2 + 15\hat{I}_1^{[0,2]}(2,2,2)\\
&- 15\hat{I}_2^{[0,2]}(2,2,2) + 10\hat{I}_0^{[0,1]}(3,1,1) + 15\hat{I}_0^{[0,1]}(2,2,1) - 5\hat{I}_1^{[0,3]}(3,2,2)\\
&+ 5\hat{I}_2^{[0,3]}(3,2,2) - 5\hat{I}_2(3,2,2)m_c^6 - 5\hat{I}_0(3,2,2)m_c^6\\
&+ 5\hat{I}_1(3,2,1)m_c^4 - 10\hat{I}_0(3,2,1)m_c^4 - 15\hat{I}_2(4,1,1)m_c^4 - 15\hat{I}_2(2,2,2)m_c^4\\
&+ 15\hat{I}_1(2,2,2)m_c^4 + 15\hat{I}_1(4,1,1)m_c^4 + 15\hat{I}_2^{[0,1]}(3,2,2)m_c^4 - 15\hat{I}_1^{[0,1]}(3,2,2)m_c^4\\
&- 15\hat{I}_0(2,2,2)m_c^4 + 5\hat{I}_2(3,2,2)m_c^5 m_b + 5\hat{I}_0(3,2,2)m_c^5 m_b - 5\hat{I}_1(3,2,2)m_c^5 m_b\\
&+ 5\hat{I}_0(3,2,2)m_c^4 m_b^2 - 5\hat{I}_1(3,2,2)m_c^4 m_b^2 + 5\hat{I}_1(3,2,2)m_c^3 m_b^3 + 5\hat{I}_1(3,2,2)m_c^6\\
&- 5\hat{I}_1(3,1,2)m_c^3 m_b + 10\hat{I}_1^{[0,1]}(3,2,2)m_c^3 m_b + 10\hat{I}_1(2,3,1)m_c^3 m_b\\
&- 10\hat{I}_0^{[0,1]}(3,2,2)m_c^3 m_b + 5\hat{I}_0(3,1,2)m_c^3 m_b - 5\hat{I}_1(3,2,1)m_c^3 m_b\\
&- 10\hat{I}_1(2,2,2)m_c^3 m_b + 5\hat{I}_2(3,1,2)m_c^3 m_b + 15\hat{I}_2(4,1,1)m_c^3 m_b\\
&+ 10\hat{I}_0(3,2,1)m_c^3 m_b - 10\hat{I}_2^{[0,1]}(3,2,2)m_c^3 m_b + 5\hat{I}_2^{[0,1]}(3,2,2)m_b^4\\
&- 15\hat{I}_0(4,1,1)m_c^4 + 5\hat{I}_1(3,1,2)m_c^4 - 10\hat{I}_2(3,1,2)m_c^4\\
&- 5\hat{I}_0(3,1,2)m_c^4\hat{I}_0(2,2,2)m_c^3 m_b + 5\hat{I}_2(3,2,1)m_c^3 m_b - 10\hat{I}_2(2,3,1)m_c^3 m_b\\
&- 10\hat{I}_2(2,2,2)m_c^2 m_b^2 - 20\hat{I}_2(3,2,1)m_c^2 m_b^2 + 30\hat{I}_0(1,4,1)m_c^2 m_b^2\\
&+ 10\hat{I}_1(3,2,1)m_c^2 m_b^2 - 5\hat{I}_0(3,2,1)m_c^2 m_b^2 + 10\hat{I}_2^{[0,1]}(3,2,2)m_c^2 m_b^2\\
&+ 30\hat{I}_2(1,4,1)m_b^4 - 15\hat{I}_2(4,1,1)m_c^2 m_b^2 + 30\hat{I}_2(1,4,1){m_c}^2 m_b^2 + 5\hat{I}_2(3,2,1)m_c m_b^3\\
&+ 5\hat{I}_1^{[0,1]}(3,2,2)m_c {m_b}^3 + 5\hat{I}_2(3,1,2)m_c m_b^3 - 30\hat{I}_2(1,4,1)m_c {m_b}^3
\end{aligned}
$$

$$- 10\hat{I}_2(2,3,1)m_c m_b^3 - 5\hat{I}_2(3,2,1)m_b^4 - 30\hat{I}_0(1,4,1)m_c m_b^3$$
$$+ 10\hat{I}_0(2,3,1)m_c m_b^3 + 10\hat{I}_2^{[0,1]}(2,3,1)m_c m_b - 40\hat{I}_0(1,3,1)m_c m_b$$
$$+ 15\hat{I}_0(2,1,2)m_c m_b - 10\hat{I}_2(2,2,2)m_b^4$$

where

$$\hat{I}_n^{[i,j]}(a,b,c) = \left(M_1^2\right)^i \left(M_2^2\right)^j \frac{d^i}{d\left(M_1^2\right)^i} \frac{d^j}{d\left(M_2^2\right)^j} \left[\left(M_1^2\right)^i \left(M_2^2\right)^j \hat{I}_n(a,b,c)\right].$$

References

[1] Drutskoy A *et al* (Belle Collaboration) 2005 *Phys. Rev. Lett.* **96** 061802
[2] Aubert B *et al* (BaBar Collaboration) 2003 *Phys. Rev. Lett.* **90** 242001
[3] Besson D *et al* (CLEO Collaboration) 2003 *Phys. Rev.* D **68** 032002
[4] Nikami Y *et al* (Belle Collaboration) 2004 *Phys. Rev. Lett.* **92** 012002
[5] Krokovny P *et al* (Belle Collaboration) 2003 *Phys. Rev. Lett.* **91** 262002
[6] Aubert B *et al* (Babar Collaboration) 2004 *Phys. Rev. Lett.* **93** 181807
[7] Aubert B *et al* (Babar Collaboration) 2004 *Phys. Rev.* D **69** 031101
[8] Aubert B *et al* (Babar Collaboration) 2004 arXiv:hep-ex/0408067
[9] Colangelo P, De Fazio F and Ferrandes R 2004 *Mod. Phys. Lett.* A **19** 2083
[10] Swanson E S 2006 *Phys. Rep.* **429** 243
[11] Colangelo P, De Fazio F and Ozpineci A 2005 *Phys. Rev.* D **72** 074004
[12] Godfrey S and Isgur N 1985 *Phys. Rev.* D **32** 189
[13] Godfrey S and Kokoski R 1991 *Phys. Rev.* D **43** 1679
[14] Di Pierro M and Eichten E 2001 *Phys. Rev.* D **64** 114004
[15] Close F E and Swanson E S 2005 *Phys. Rev.* D **72** 094004
[16] Godfrey S 2003 *Phys. Lett.* B **568** 254
[17] Bardeen W A, Eichten E J and Hill C T 2003 *Phys. Rev.* D **68** 054024
[18] Nowak M A, Rho M and Zahed I 2004 *Acta Phys. Pol.* B **35** 2377
[19] Deandrea A, Nardulli G and Polosa A D 2003 *Phys. Rev.* D **68** 097501
[20] Cahn R N and Jackson J D 2003 *Phys. Rev.* D **68** 037502
[21] Dai Y B, Huang C S, Liu C and Zhu S L 2003 *Phys. Rev.* D **68** 114011
[22] Lucha W and Schobert F 2003 *Mod. Phys. Lett.* A **18** 2837
[23] Hofmann J and Lutz M F M 2004 *Nucl. Phys.* A **733** 142
[24] Sadzikowski M 2004 *Phys. Lett.* B **579** 39
[25] Becirevic D *et al* 2004 *Phys. Lett* B **599** 59
[26] Lee T, Lee I W, Min D P and Park B Y 2007 *Eur. Phys. J.* C **49** 737
[27] Kolomeitsev E and Lutz M 2004 *Phys. Lett.* B **582** 39
[28] Barnes T, Close F E and Lipkin H J 2003 *Phys. Rev.* D **68** 054006
[29] Cheng H Y and Hou W S 2003 *Phys. Lett.* B **566** 193
[30] Terasaki K 2003 *Phys. Rev.* D **68** 011501
[31] Browder T E, Pakvasa S and Petrov A A 2004 *Phys. Lett.* B **578** 365
[32] Dmitrasinovic U 2004 *Phys. Rev.* D **70** 096011
Dmitrasinovic U 2005 *Phys. Rev. Lett.* **94** 162002
[33] Bracco M E, Lozea A, Matheus R D, Navarra F S and Nielsen M 2005 *Phys. Lett.* B **624** 217
[34] Kim H and Y Oh 2005 *Phys. Rev.* D **72** 074012
[35] Szczepaniak A P 2003 *Phys. Lett.* B **567** 23
[36] Thomas C E 2006 *Phys. Rev.* D **73** 054016
[37] Cheng H Y 2003 *Phys. Rev.* D **68** 094005
[38] Azizi K, Khosravi R and Bashiry V 2008 *Eur. Phys. J.* C **56** 357
[39] Azizi K and Khosravi R 2008 *Phys. Rev.* D **78** 036005
[40] Azizi K, Falahati F, Bashiry V and Zebarjad S M 2008 *Phys. Rev.* D **77** 114024
[41] Deshpande N G, Trampetic J and Panose K 1989 *Phys. Rev.* D **39** 1461
Lim C S, Morozumi T and Sanda A I 1989 *Phys. Lett.* B **218** 343
[42] Ghahramany N, Khosravi R and Azizi K 2008 *Phys. Rev.* D **78** 116009
[43] Geng C Q, Hwang C W and Liu C C 2002 *Phys. Rev.* D **65** 094037
[44] Breub C, Ioannissian A and Wyler D 1995 *Phys. Lett.* B **346** 149

[45] Geng C Q and Kao C P 1996 *Phys. Rev.* D **54** 5636
[46] Shifman M A, Vainshtein A I and Zakharov V I 1979 *Nucl. Phys.* B **147** 385
[47] Ceccucci A, Ligeti Z and Sakai Y 2006 PDG *J. Phys. G: Nucl. Part. Phys.* **33** 139
[48] Buras A J and Muenz M 1995 *Phys. Rev.* D **52** 186
[49] Bashiry V and Azizi K 2007 *J. High Energy Phys.* JHEP07(2007)064 JHEP07(2007)064
[50] Veseli S and Dunietz I 1996 *Phys. Rev.* D **54** 6803
[51] Colangelo P, Nardulli G and Paver N 1993 *Z. Phys.* C **57** 43
[52] Kiselev V V and Tkabladze A V 1993 *Phys. Rev.* D **48** 5208
[53] Aliev T M and Yilmaz O 1992 *Nuovo Cimento* A **105** 827
[54] Amsler C *et al* 2008 Particle data group *Phys. Lett.* B **667** 1
[55] Colangelo P, De Fazio F and Ozpineci A 2005 *Phys. Rev.* D **72** 074004
[56] Aliev T M and Savci M 1998 *Phys. Lett.* B **434** 358